Performance Analysis of Thermoelectric Pellets with Non-Constant Cross Sections

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Abstract

Thermoelectricity theory has stated that both, heat absorbed and rejected at the cold and hot sides of a thermoelectric pair by Peltier effect only depends on the thermoelectric properties of the semiconductors, the absolute temperature in the unions, and the electrical current through the pair. However, the irreversible phenomena, Joule and Fourier effect, are volumetric effects. That is to say, they can be affected by the geometry of the pellets, and furthermore, they have an influence on the temperature at the surfaces where the Peltier effect takes place. As consequence, the net heat power pumped by a thermoelectric pair can be modified when non-constant cross section pellets are used.

In this paper, variable cross section pellets are studied, analysing the influence on the heat power absorbed, and on the coefficient of performance. Firstly the study of the effect just on the irreversible phenomena, and secondly, taking into account the influence on the irreversible effects together with Peltier effect.

This problem has been conveniently studied fixing the volume of the pellets and when the lateral surfaces of pellet are not adiabatic. In this paper the problem has been analysed on a different way. Different geometrical characteristics of the pellets are analysed in order to improve the exchange of heat power at the ends of the pellets evaluating the effect produced in other variables, specially in the volume of the pellets. The lateral surfaces are considered adiabatic taking into account the efforts done by the manufactures of commercial thermoelectric modules to reduce these thermal losses. The results are compared with typical values obtained using constant cross sections.

The conclusions obtained in this work can be of interest in the design of thermoelectric pairs in applications where the size of the pellets allows for the use of variable cross sections, or the thermolectric properties are extremely sensitive to temperature variations.

Introduction

Being a(x) the function which defines the cross section variation of the pellet respect to the axis where the thermal gradient takes place, the optimum a(x) will minimise the effect of irreversible phenomena under the same functional conditions: temperature difference (ΔT), electrical current (I); material properties, and volume of the pellet.

In this paper the influence of this function a(x) is studied in two ways: firstly considering just only the irreversible phenomena in the pellet: Joule and Fourier effect; and secondly taking also into account the Peltier effect.

The study was carried out using an element with the geometry shown in figure 1, and the following characteristics:

- The electrical current flows along the x axis.
- There is a thermal gradient between its ends ($T_h=T_{x=L}$ and $T_c=T_{x=0}$).
- Its cross section varies along the x axis following the function a(x).
- The lateral surfaces are considered adiabatic.
- The problem is considered one-dimensional, that is, temperature is constant in all the points of a same section, T(x)=T.
- Isotropic material with constant thermoelectric properties: ρ, λ, and σ.

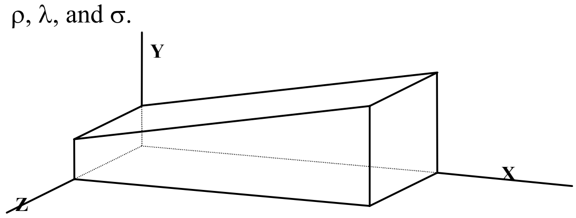

Figure 1. Scheme of the analysed pellet

1. Influence of the pellet geometry on the irreversible phenomena

The temperature distribution along x, when Fourier and Joule effect are studied, is given by the following differential equation, [1].

$$\frac{d^2T}{dx^2}+\frac{1}{a}\cdot\frac{da}{dx}\cdot\frac{dT}{dx}+\frac{\rho}{\lambda}\cdot\frac{I^2}{a^2}=0 \qquad \{1\}$$

Equation {1} can be expressed using the variable change μ=dT/dx as:

$$(\mu\cdot a)_x-(\mu\cdot a)_0=-\frac{\rho\cdot I^2}{\lambda}\cdot\int_0^x\frac{dx}{a} \qquad \{2\}$$

The most important geometric parameter of a pellet is the ratio between its length and its cross section, E=L/A, called from now "slimness". This concept can be extended for the case of a non-constant cross section thermoelement in the following way:

$$e=e(x)=\int_o^x\frac{dx}{a(x)} \qquad \{3\}$$

and then

$$de=\frac{dx}{a(x)} \qquad E=\int_0^L de \qquad \{4\}$$

Using {3}, equation {2} can be rewritten as:

$$(\mu \cdot a)_x - (\mu \cdot a)_0 = -\frac{\rho \cdot I^2}{\lambda} e \qquad \{5\}$$

The thermal power between two sections a(x) and a(x+dx) at temperatures T and T+dT respectively, is:

$$\dot{Q} = -\lambda \cdot a \cdot \frac{dT}{dx} = -\lambda \cdot a \cdot \mu \qquad \{6\}$$

then, the product μa can be expressed as a function of the thermal power:

$$\mu \cdot a = -\frac{\dot{Q}}{\lambda} \qquad \{7\}$$

being

$$(\mu \cdot a)_0 = -\left.\frac{\dot{Q}}{\lambda}\right)_0 = -\frac{\dot{Q}_0}{\lambda} \text{ and } (\mu \cdot a)_x = -\frac{\dot{Q}(x)}{\lambda} \qquad \{8\}$$

Substituting {8} in {2}

$$dT = -\frac{\rho \cdot I^2}{\lambda} \cdot \frac{e}{a} \cdot dx + \frac{\dot{Q}_o}{a \cdot \lambda} \cdot dx \qquad \{9\}$$

Integrating equation {9}, the temperature distribution along the pellet is obtained. As a=a(x) is unknown, equation {3} can be substituted in {9} getting the following equation:

$$dT = -\frac{\rho \cdot I^2}{\lambda} \cdot e \cdot de + \frac{\dot{Q}_o}{\lambda} \cdot de \qquad \{10\}$$

Equation {10} can be integrated between e=0 (corresponding to x=0, T=T_c) and e=E (corresponding to x=L, T=T_h) obtaining that:

$$T_h = T_c - \frac{1}{2} \cdot E^2 \cdot I^2 \cdot \frac{\rho}{\lambda} + \frac{\dot{Q}_0}{\lambda} \qquad \{11\}$$

reordering equation {11}

$$\frac{\dot{Q}_0}{\lambda} = \frac{\Delta T}{E} + \frac{1}{2} \cdot \frac{\rho}{\lambda} \cdot E \cdot I^2 \qquad \{12\}$$

and substituting {12} in {9}, the following expression is obtained:

$$dT = -\frac{\rho \cdot I^2}{2 \cdot \lambda} \cdot \frac{2 \cdot e - E}{a} \cdot dx + \frac{\Delta T}{a \cdot E} \cdot dx \qquad \{13\}$$

The temperature distribution as a function of e(x) is obtained integrating equation {13} between x=0 and x (generic distance):

$$T = -\frac{\rho \cdot I^2}{2} \cdot \frac{e^2 - e \cdot E}{\lambda} + \frac{\Delta T \cdot e}{E} + T_c \qquad \{14\}$$

Using {6}, the thermal power which crosses any section would be:

$$\dot{Q} = \frac{\rho \cdot I^2}{2} \cdot (2 \cdot e - E) - \Delta T \cdot \frac{\lambda}{E} \qquad \{15\}$$

In {15}, the first term on the right represents the thermal power by Joule effect ($\dot{Q}_J$), while the second term is due to Fourier effect ($\dot{Q}_F$). From now, the thermal power expressed in {15} is designated as $\dot{Q}_{JF}$.

Notice that both, the temperature distribution and the thermal power do not depend directly on the function a(x), although it is included in the definition of e(x). The thermal power interchanged with the ambient at x=0, where e=0 is:

$$\dot{Q}_{JF}\big|_{x=0} = -\frac{\rho \cdot I^2}{2} \cdot E - \Delta T \cdot \frac{\lambda}{E} \qquad \{16\}$$

This thermal power is minimised when the geometry of the pellet is:

$$E_{opt} = \frac{1}{I} \cdot \sqrt{\frac{2 \cdot \lambda \cdot \Delta T}{\rho}} \qquad \{17\}$$

and the minimum thermal power, with this geometry, at x=0 would be:

$$\dot{Q}_{JF_{\min}}\big|_{x=0} = -I \cdot \sqrt{2 \cdot \rho \cdot \lambda \cdot \Delta T} \qquad \{18\}$$

The thermal power at x=L, where e=E is

$$\dot{Q}_{JF}\big|_{x=L} = \frac{\rho \cdot I^2}{2} \cdot E - \Delta T \cdot \frac{\lambda}{E} \qquad \{19\}$$

This function is monotonous increasing respect to E. The thermal power at x=L increase with E due to the raise in the generation of thermal power by Joule effect.

Comparing equations {16} and {19} the following conclusions are obtained:

- The total slimness E is the unique geometric variable which influences on the value of the thermal power at the ends of the pellet.
- The thermal power by Fourier effect has the same value and sign ($-\Delta T \cdot \lambda / E$) at both ends, (flows to the inner part at x=L and to the outside at x=0).
- The thermal power by Joule effect has the same value ($\rho \cdot I^2 / 2 \cdot E$) at both ends, being the middle of the total power generated by Joule effect inside the pellet, independently of the geometry, and if there is or not a symmetry respect to the ends of the thermoelement. Notice that the thermoelectric properties of the pellets are considered constant with temperature, and the model is one-dimensional, and these are the conditions to obtain this result.
- Using the value of E which optimises the thermal power at x=0, the thermal power at x=L is null. In this case the thermal powers due to Joule and Fourier effects achieve the same absolute value but with different signs at x=L.

2. Influence of the pellet geometry on Peltier effect and irreversible phenomena at the same time

In a thermoelectric pair, the Peltier effect is always joined to the irreversible phenomena. In this section, the influence of the geometry of the pellet on the net value of the thermal powers, and the cooling efficiency is analysed.

2.1. Influence on the thermal powers

The thermal power at the ends of a pellet considering the Peltier effect is given by the following expressions:

$$\dot{Q}_{x=0} = \dot{Q}_P\big|_{x=0} + \dot{Q}_{JF}\big|_{x=L} = \sigma_0 \cdot I \cdot T_0 - \frac{\rho \cdot I^2}{2} \cdot E - \frac{\lambda}{E} \cdot \Delta T \qquad \{20\}$$

$$\dot{Q}_{x=L} = \dot{Q}_P\big|_{x=L} + \dot{Q}_{JF}\big|_{x=L} = \sigma_L \cdot I \cdot T_L + \frac{\rho \cdot I^2}{2} \cdot E - \frac{\lambda}{E} \cdot \Delta T \qquad \{21\}$$

Taking into account that $\dot{Q}_P$ is independent from E, E_{opt} optimises also $\dot{Q}_{x=0}$ being the maximum:

$$\dot{Q}_{\max}\Big|_{x=0} = \sigma \cdot I \cdot T_0 - I \cdot \sqrt{2} \cdot \sqrt{\rho \cdot \lambda \cdot \Delta T} \qquad \{22\}$$

By the same reason, $\dot{Q}_{x=L}$ does not have neither a maximum nor a minimum respect to E, as it was commented in the previous section.

2.2. Influence on the cooling efficiency

The cooling efficiency in the pellet is expressed by the following expression:

$$\Phi = \frac{\dot{Q}_{x=0}}{\dot{W}} = \frac{\dot{Q}_{x=0}}{\dot{Q}_{x=L} - \dot{Q}_{x=0}} = \frac{\sigma_0 \cdot I \cdot T_0 - \rho \cdot E \cdot \frac{I^2}{2} - \lambda \cdot \frac{\Delta T}{E}}{I \cdot (\sigma_L \cdot T_L - \sigma_0 \cdot T_0) + \rho \cdot I^2 \cdot E} \qquad \{23\}$$

Analysing equation {23} respect to E, the value which optimises the efficiency is given by:

$$E_{\Phi_{max}} = \frac{\lambda}{\sigma} \cdot \frac{\Delta T}{T_m} \cdot \frac{1}{I} \cdot \left(1 + \sqrt{1 + z \cdot T_m}\right) \qquad \{24\}$$

being the maximum efficiency

$$\phi_{max} = \frac{T_0}{\Delta T} \cdot \frac{\left(1 + \sqrt{1 + z \cdot T_m}\right) \cdot \left(1 - \frac{\Delta T}{T_0 \cdot z \cdot T_m}\right) - \left(1 + \frac{\Delta T}{T_0}\right)}{\left(1 + \sqrt{1 + z \cdot T_m}\right) \cdot \left(1 + \frac{2}{T_m}\right)} \qquad \{25\}$$

In the case studied, with adiabatic lateral surfaces, the maximum efficiency is independent from both, the functions e(x) and a(x). It is only a function of the total E; the temperatures at the ends of the pellet; and the thermoelectric properties, supposed constant. This conclusion contradicts at first view the result obtained by Thacher in [2]. This author studied the entropy generated inside a pellet, and deducted a expression for the efficiency as a function of the entropy. Applying variational calculation to that expression, Thacher demonstrated that the maximum efficiency is achieved when the cross section of the pellet is constant (a(x)=cte).

However, this unreal contradiction between Thacher´s conclusion and the results stated in this article roots in the fact that Thacher includes the hypothesis of considering all the thermoelements with the same volume, that is:

$$\int_0^L a(x) \cdot dx = cte \quad \text{for any a(x) function}$$

This restriction is not included in this study. So, the main conclusion obtained in this section is that the maximum efficiency can be obtained with any geometry whose total slimness E was equal to the value obtained in {24}. It is clear that each geometry will have a different volume. In the next section, it will be proved that the minimum volume correspond with the function a(x)=cte.

Notice also, that the relation between the value of E which optimises the cooling power and the cooling efficiency is independent of the electrical current, it is just only a function of the thermoelectric properties, and the temperatures at the end of the pellets.

$$\frac{E_{opt}}{E_{\phi}} = \frac{\sqrt{2 \cdot z}}{\sqrt{\Delta T} \cdot \left(1 + \sqrt{1 + z \cdot \left(T_0 + \frac{\Delta T}{2}\right)}\right)} \qquad \{26\}$$

3. Analysis of pellet geometry

The idea of optimum geometry of a pellet must take into account the interaction between the pellets and the rest of elements which compound a thermoelectric module. For this reason, it is necessary to consider other "tertiary factors" of the geometry of the pellets, considered in the previous formulations, and which can influence, direct or indirectly, on the values of the efficiency, and the thermal power interchanged between the cold and hot sides of the module. These tertiary factors are the following:

- **Volume** of the pellet. Its value must be reduced, taking into account the singularity of the materials used, and the complex processes to obtain them.
- **Contact surfaces** with the electric bridges. Peltier effect takes place on this surfaces. The value of the area influence on the temperature and as a consequence in the thermal power generated or absorbed by Peltier effect. Hence, an additional thermal power is generated in the electrical bridges by Joule effect, and the thermal resistance (Fourier effect), which influence on heat pumping, can vary depending on the design of the pellet.
- The **structural resistance** of the pellet. Not all the shapes of the pellets and configurations are valid from the point of view of obtaining robust thermoelectric modules, which support mechanical efforts.
- **Fabrication cost**. This cost can be affected by the proper geometry, although it will depend on the size of the series fabricated.
- **The distance between the heat source and the heat sink**. This distance influence on the thermal bypass through the material which insulates the free space between the pellets.
- **Maximum temperature inside the pellet**. This temperature can be higher than the temperature at the ends of the pellets. This can be positive or negative depending on the behaviour of the material properties (thermal conductivity, λ and electrical resistivity, ρ) with the temperature. If dλ/dT<0 or dρ/dT<0, an increment in temperature improves the thermoelectric characteristics of the pellet, and as consequence, for a fixed E, an electrical current higher than:

  $$I_{opt} = \frac{1}{E} \cdot \sqrt{\frac{2 \cdot \lambda \cdot \Delta T}{\rho}}$$

  can maintain and even improve the value of $\dot{Q}_{c_{max}}$. The opposite effect can be produced if dλ/dT>0 or dρ/dT>0.

3.1. Geometry with minimum volume

Being the infinitesimal volume dv=a(x)dx, the total volume of the pellet would be:

$$V = \int_0^L a(x) \cdot dx$$

Using the criteria established in the previous section, the total slimness E and the length of the pellet L are fixed in order to reduce the thermal power transferred from the hot to the cold side through the material which imbibed the pellets.

However, the two conditions previously mentioned can be satisfied by any function a(x), obtaining infinite volumes for a pellet. In this section, the function a(x) which minimises the total volume of the pellet is deducted.

Being a(x) the function which defines the generic area of a pellet (continuous and derivable and with existence ($0<a(x)<\infty$) in the interval $0\leq x\leq L$), see figure 2, and L the length of the pellet, the ratio E and the total volume can be expressed as:

$$E = \int_0^L u(x)\cdot dx \quad \{27\}$$

$$V = \int_0^L a(x)\cdot dx \quad \{28\}$$

where u(x)=1/a(x)

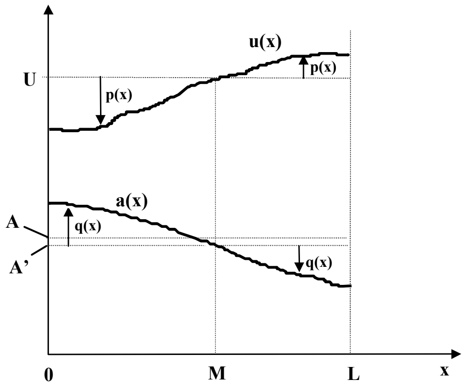

Figure 2. Generic functions: a(x) and u(x)

If U is the mean value of u(x) in the interval $0\leq x\leq L$,

$$U = \frac{1}{L}\cdot\int_0^L u(x)\cdot dx = \frac{E}{L}$$

and x_0=M is the value of the abscissa where u(M)=U, the straight line x=M intersect the function a(x) in the point (M,A') so that A'=a(M)=1/u(M)=1/U. Then, it is possible to define the functions:

$$q(x) = a(x) - A' \qquad p(x) = u(x) - U$$

and the total volume of the pellet would be:

$$V = \int_0^L [A'+q(x)]\cdot dx = A'\cdot L + \int_0^L q(x)\cdot dx \quad \{29\}$$

furthermore

$$\int_0^L q(x)dx = \int_0^L [a(x)-A']dx = \int_0^L\left[\frac{1}{u(x)}-\frac{1}{U}\right]dx = -\frac{1}{U}\cdot\int_0^L \frac{p(x)}{u(x)}dx \quad \{30\}$$

substituting equation {30} in {29}

$$V = \int_0^L a(x)\cdot dx = A'\cdot L - \frac{1}{U}\cdot\int_0^L \frac{p(x)}{u(x)}\cdot dx \quad \{31\}$$

The integral on the right term of {31} can be expressed as:

$$\int_0^L \frac{p(x)}{u(x)}\cdot dx = \int_0^L\left(1-\frac{U}{u(x)}\right)\cdot dx = L - \int_0^L \frac{U}{u(x)}\cdot dx \quad \{32\}$$

using the Cauchy-Schwartz inequality, it is possible to demonstrate that:

$$\int_0^L \frac{U}{u(x)}\cdot dx \geq L \quad \text{and then} \quad \int_0^L \frac{p(x)}{u(x)}\cdot dx \leq 0 \quad \{33\}$$

and substituting {33} in {31} the following expression for the volume of the pellet is found:

$$V = \int_0^L a(x)\cdot dx \geq A'\cdot L \quad \{34\}$$

Equation {34}means that V=A'L is the minimum value of the volume of any pellet with a length L and a slimness E. This minimum volume correspond with p(x)=0. In this case, a(x)=A', that is, a pellet with a constant cross section. The condition for minimum volume for given values of L and E is a pellet with a prismatic constant cross section (a(x)=L/E=A').

This assumption allows for demonstrating that $A'\leq A$, see figure 2, being A

$$A = \frac{1}{L}\cdot\int_0^L a(x)\cdot dx$$

and the value of A is equal to A' when a(x)=A.

3.2. Maximum temperature inside the pellet

The temperature distribution along the pellet is given by differential equation {13}. Equalling to zero dT/dx, the value of the function e(x) which maximises the temperature inside the pellet would be:

$$e_{T_{max}} = \frac{E}{2} + \frac{\Delta T}{E}\cdot\frac{\lambda}{\rho\cdot I^2} \quad \{35\}$$

and the maximum temperature :

$$T_{max} = T_c + \left(\frac{E}{2}+\frac{\Delta T}{E}\cdot\frac{\lambda}{\rho\cdot I^2}\right)\cdot\left(\frac{\Delta T}{2\cdot E}+\frac{\rho\cdot I^2}{2\cdot\lambda}\cdot E\right) \quad \{36\}$$

Analysing equation {36}, the maximum temperature value inside the pellet depends on:

- The electrical current, I
- Temperatures at the ends of the pellet, T_c, and T_h, or the temperature difference, ΔT.
- The thermoelectric properties of the material, ρ and λ.
- The value of E.

So, geometrically distinct pellets will have the same maximum temperature if the rest of characteristics (I, ΔT, T_c, ρ and λ) have the same value, although the function e=e(x) was different. Remember that L and E are fixed. The point inside the pellet where the temperature is maximum will have the same value of $e_{T_{max}}$ for any geometry, although the coordinate x will be different.

However, the temperature distribution in the thermoelement is affected by its geometry a=a(x), and as consequence the performance of the pellet, taking into account that the thermoelectric properties in fact are dependent on temperature.

4. Detailed study of some geometries

In spite of some characteristics of the performance of the pellet are independent on the specific geometry (a(x)), just only depend on the total value of E. It is profitable to analyse the behaviour of some specific geometries, which can improve the results obtained with constant cross-section pellets, taking into account some tertiary factors. In this work, the tertiary factor considered was the area at the ends of the pellet. The highest areas, the best heat dissipation of the thermal powers, and as consequence, lower temperature difference in the pellet will be achieved. The geometries studied maintained volumes close to the minimum value but increasing the areas at the ends with the same values of L an E of the constant cross section.

Four different geometries were analysed and compared:

- Constant cross-section: a(x)=A
- Linear variation of the cross-section:
  $$a(x)=\left|A+H\cdot(M-x)\right|_{0\le x\le M}+\left|A+G\cdot(x-M)\right|_{M\le x\le L}$$
- Quadratic variation of the cross section
  $$a(x)=\left|A+H\cdot(M-x)^2\right|_{0\le x\le M}+\left|A+G\cdot(x-M)^2\right|_{M\le x\le L}$$
- Exponential variation of the cross-section
  $$a(x)=\left|A+H^{C\cdot(M-x)}\right|_{0\le x\le M}+\left|A+G^{B\cdot(x-M)}\right|_{M\le x\le L}$$

The parameters A, G, and H must be positive, 0≤M≤L and with values such as E and L have the same value in all cases.

All the geometries studied have a contraction located at a certain distance (M) from the origin, the area of this contraction is A, see figure 3.

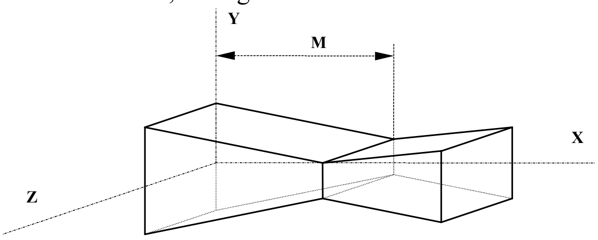

Figure 3. Scheme of non-constant cross section pellet

The mean thermoelectric properties used were σ=200μV/K, λ=1.59 W/mK, and ρ=0,00887 Ωmm. The length of the pellet was L=1,6 mm and E=0,816 mm^{-1}; the temperature at the cold side T_c=273 K, the temperature at the hot side T_h=293 K, ΔT=20 K.

The pellets were analysed working with three different values of electrical current: current which optimises the cooling power, $I_1=I_{\dot{Q}_c}\Big|_{max}=3{,}281\ \mathrm{A}$; current which optimises the cooling efficiency, $I_2=I_{\phi}\Big|_{max}=1{,}613\ \mathrm{A}$; and $I_3=I^*=2\cdot I_{\dot{Q}_c}\Big|_{max}=6{,}563\ \mathrm{A}$.

Table 1. Thermal powers at the ends of the pellet

		I_1, A	I_2, A	I_3, A
		3,2815	1,6130	6,5631
$Q_{Px=0}$	W	0,1792	0,0881	0,3583
$Q_{Fx=0}$	W	-0,0390	-0,0390	-0,0390
$Q_{Jx=0}$	W	-0,0390	-0,0094	-0,1559
$Q_{x=0}$	W	**0,1012**	**0,0397**	**0,1635**
$Q_{Px=L}$	W	0,1923	0,0945	0,3846
$Q_{Fx=L}$	W	-0,0390	-0,0390	-0,0390
$Q_{Jx=L}$	W	0,0390	0,0094	0,1559
$Q_{x=L}$	W	**0,1923**	**0,0650**	**0,5015**
QF	W	-0,0390	-0,0390	-0,0390
QJ	W	0,0779	0,0188	0,3118
ϕ		1,1116	1,5695	0,4837

Due to all the assumptions, the thermal powers, and the cooling efficiency will have the same values for all the geometries, see table 1.

4.1. Constant cross section

In this case, $a(x)=a_{x=0}=a_{x=L}$=A=cte, the function e is given by e=x/A, and the volume by V=Ax. Considering the complete pellet (x=L), e=L/A=E and V=AL, and taking into account the fixed values, A=1,96 mm^2 and V=3,136 mm^3.

Applying equation {14}, the temperature distribution along the pellet is given by:

$$T(x)=T_c+\frac{\rho\cdot I^2}{2\cdot\lambda}\cdot E^2\cdot\left(\frac{x}{L}-\frac{x^2}{L^2}\right)+\frac{x}{L}\cdot\Delta T \qquad \{37\}$$

Temperature difference along the pellet working with the three electrical currents mentioned above is shown in figure 4. Notice, that for I^*, the maximum temperature inside the pellet ($T_{max}=T_c$+31,6 K) is higher than the temperature at the hot side end ($T_h=T_c$+20 K).

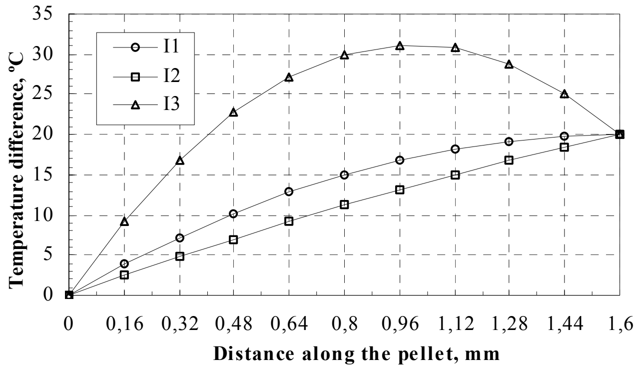

Figure 4. Temperature difference along the pellet (T_h-T_c) for different electrical currents, Tc=273 K, constant cross section.

4.2. Linear variation of the cross section

This geometry is modelled by the following equation:

$$a(x)=\left|A+H\cdot(M-x)\right|_{0\le x\le M}+\left|A+G\cdot(x-M)\right|_{M\le x\le L} \qquad \{38\}$$

The inverse of the geometric factor is:

$$E=\frac{1}{H}\cdot\left[Ln(H\cdot M+A)-LnA\right]+\frac{1}{G}\cdot\left[Ln(G\cdot(L-M)+A)-LnA\right] \qquad \{39\}$$

The cross sections at the ends of the pellets:

$$A_0=H\cdot M+A \qquad A_L=A+G\cdot(L-M) \qquad \{40\}$$

with the following limit values:

$A_0\Big|_{min}=A$ for M=0; $A_0\Big|_{max}=A+H\cdot L$ for M=L

$A_L\Big|_{min}=A$ for M=L; $A_L\Big|_{max}=A+G\cdot L$ for M=0

The volume of the pellet is given by:

$$V=\frac{1}{2}\cdot H\cdot M^2+A\cdot L+\frac{G}{2}\cdot(L-M)^2 \qquad \{41\}$$

whose limits are:

$$V_{min}=A\cdot L+L^2\cdot\frac{H\cdot G^2+G\cdot H^2}{2\cdot(H+G)^2} \text{ for } M=L\cdot\frac{G}{G+H}$$

$V_{max}=A\cdot L+\frac{H\cdot L^2}{2}$ or $V_{max}=A\cdot L+\frac{G\cdot L^2}{2}$ for M=L or M=0, if H or G is higher respectively.

If H=G

$V_{min}=A\cdot L+\frac{G\cdot L^2}{4}$ for $M=\frac{L}{2}$ and

$V_{max}=A\cdot L+\frac{G\cdot L^2}{2}$ in the two ends for M=0, and M=L

With these geometric relations, the variation of the cross section of the pellet for different positions of the contraction (0<M<1,6 mm) taking H=G=1 is shown in figure 5 and the variation of the pellet volume respect to M in figure 6.

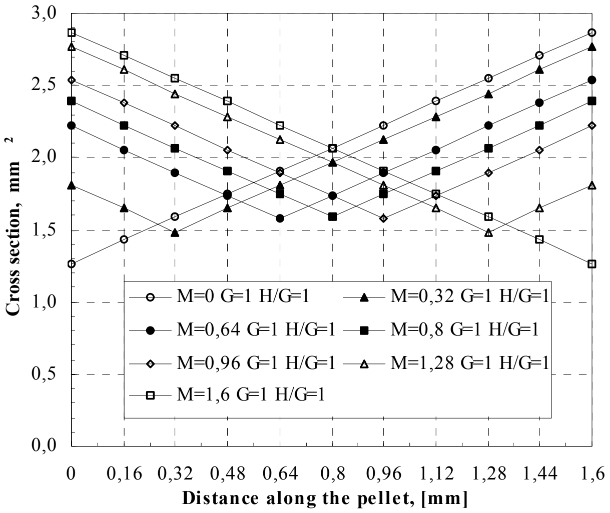

Figure 5. Cross section of the pellet when G=H=1 for different positions of the contraction, linear variation.

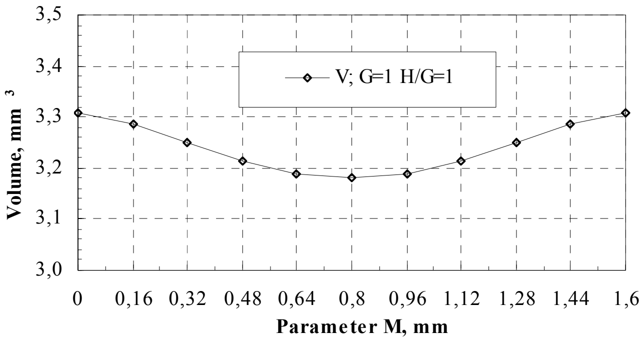

Figure 6. Volume of the pellet, linear variation.

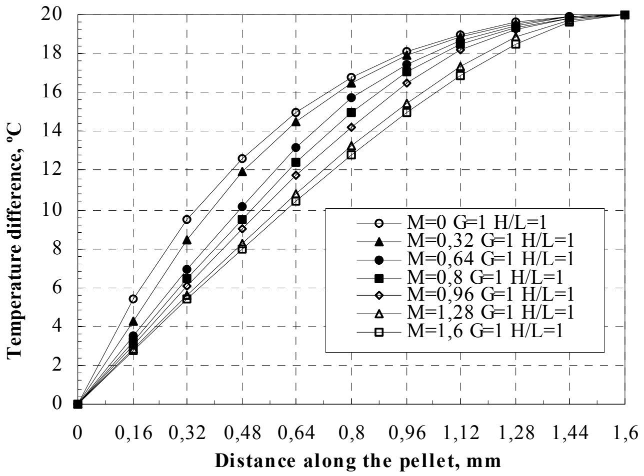

Figure 7. Temperature difference (T_h-T_c), linear variation, T_c= 273 K, G=H=1, $I_{\dot{Q}_c}\big|_{max}$

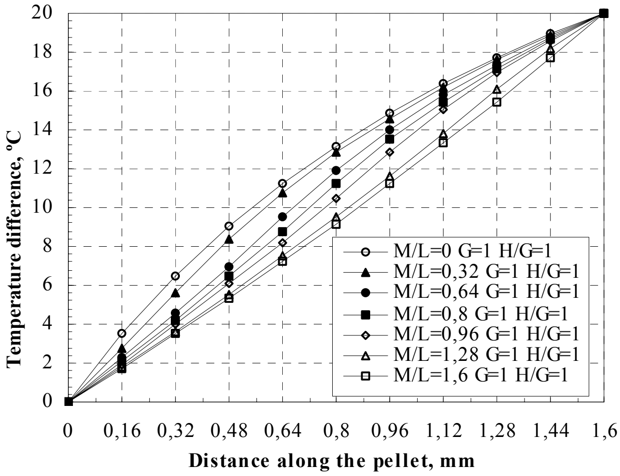

Figure 8. Temperature difference (T_h-T_c), linear variation, T_c= 273 K, G=H=1, $I_{\phi}\big|_{max}$

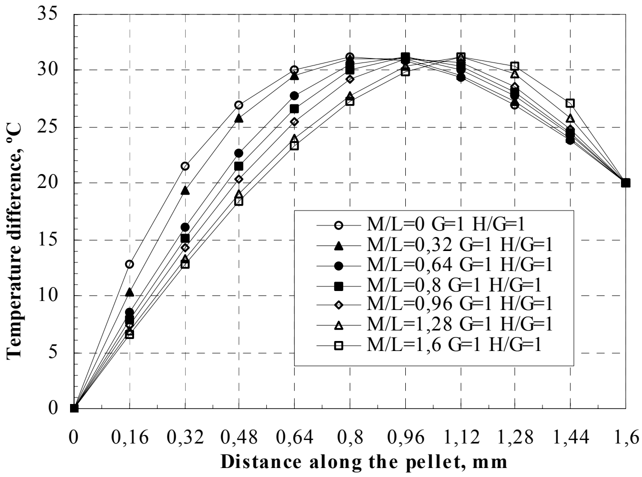

Figure 9. Temperature difference (T_h-T_c), linear variation, T_c= 273 K, G=H=1, I^*

The temperature distributions for the mentioned three values of electric currents and different contraction positions are shown in figures 7-9.

4.3. Quadratic variation of the cross section

The geometry studied is modelled by the following equation:

$$a(x) = \left|A + H\cdot(M-x)^2\right|_{0\le x\le M} + \left|A + G\cdot(x-M)^2\right|_{M\le x\le L} \quad \{42\}$$

Using the same procedure as in section 4.1, the following values were obtained:

$$E = \frac{arctg\left(\dfrac{H^{\frac{1}{2}}\cdot M}{A^{\frac{1}{2}}}\right)}{H^{\frac{1}{2}}\cdot A^{\frac{1}{2}}} + \frac{arctg\left(\dfrac{G^{\frac{1}{2}}\cdot(L-M)}{A^{\frac{1}{2}}}\right)}{H^{\frac{1}{2}}\cdot A^{\frac{1}{2}}} \quad \{43\}$$

$$A_0 = H\cdot M^2 + A \qquad A_L = A + G\cdot(L-M)^2 \quad \{44\}$$

with the following limit values:

$A_0\big|_{min} = A$ for M=0; $A_0\big|_{max} = A + H\cdot L^2$ for M=L

$A_L\big|_{min} = A$ for M=L; $A_L\big|_{max} = A + G\cdot L^2$ for M=0

$$V = \frac{1}{3}\cdot H\cdot M^3 + A\cdot L + \frac{G}{3}\cdot(L-M)^3 \qquad \{45\}$$

$$V_{min} = A\cdot L + L^3\cdot\frac{H\cdot\left(G-\sqrt{G\cdot H}\right)^3 - G\cdot\left(H-\sqrt{G\cdot H}\right)^3}{3\cdot(G-H)^3}, \text{ for } M = L\cdot\frac{G-\sqrt{G\cdot H}}{G-H}$$

$V_{max} = A\cdot L + \frac{H\cdot L^3}{3}$ or $V_{max} = A\cdot L + \frac{G\cdot L^3}{3}$ for M=L and M=0

If H or G is higher respectively. If H=G, $V_{min} = A\cdot L + \frac{G\cdot L^3}{12}$ for M=L/2; and $V_{max} = A\cdot L + \frac{G\cdot L^3}{3}$ in both ends M=0 and M=L.

4.4. Exponential variation of the cross section

The studied geometry is modeled by the following equation:

$$a(x) = \left|A + H^{C\cdot(M-x)}\right|_{0\le x\le M} + \left|A + G^{B\cdot(x-M)}\right|_{M\le x\le L} \qquad \{46\}$$

$$E = \frac{1}{A\cdot C\cdot LnH}\cdot\left(1-H^{-CM}\right) + \frac{1}{A\cdot B\cdot LnG}\cdot\left(1-G^{-B\cdot(L-M)}\right) \qquad \{47\}$$

$$A_0 = A\cdot H^{C\cdot M} \qquad A_L = A\cdot G^{B\cdot(L-M)}$$

with the following limit values:

$A_0\big|_{min} = A$ for M=0; $A_0\big|_{max} = A\cdot H^{CL}$ for M=L

$A_L\big|_{min} = A$ for M=L; $A_L\big|_{max} = A\cdot G^{BL}$ for M=0

$$V = \frac{1}{C\cdot LnH}\cdot\left(H^{CM}-1\right) + \frac{A}{B\cdot LnG}\cdot\left(G^{B\cdot(L-M)}-1\right)$$

$$V_{min} = \frac{A}{C\cdot LnH}\cdot\left(H^{\frac{BCLLnG}{CLnH+BLnG}}-1\right) + \frac{A}{B\cdot LnG}\cdot\left(G^{\frac{B\cdot CLLnH}{CLnH+BLnG}}-1\right)$$

for $M = L\cdot\frac{B\cdot LnG}{C\cdot LnH + B\cdot LnG}$

$V_{max} = \frac{A}{C\cdot LnH}\cdot\left(H^{CL}-1\right)$ or $V_{max} = \frac{A}{B\cdot LnG}\cdot\left(G^{BL}-1\right)$ for M=L or M=0 if H or G is higher respectively. If H=G, $V_{min} = \frac{2\cdot A}{B\cdot LnG}\cdot\left(G^{\frac{BL}{2}}-1\right)$ for M=L/2; and $V_{max} = \frac{A}{B\cdot LnG}\cdot\left(G^{BL}-1\right)$ in both ends M=0 and M=L.

In this case B=C=1 and G, and H were equal to number "e" (2,718).

Similar graphs to the ones represented in figures 5-9 were obtained for the quadratic and exponential variation of the cross section.

4.5 Comparative analysis of the different geometries

The most relevant characteristics of the different geometries (linear, quadratic, and exponential variation) are compared with the results for the constant cross section, considering the contraction in the middle of the pellet (M=0,8). The results are shown in figures 10-15, and geometry characteristics are summarised in table 2.

Table 2. Geometric characteristics of pellets compared.

L=1.6 mm, E=0.816 mm^{-1}	**A_M (mm^2)**	**A_0=A_L (mm^2)**	**V (mm^3)**
Constant	1.961	1.961	3.137
Linear (H=G=1, M/L=0.5)	1.538	2.388	3.181
Quadratic (H=G=1, M/L=0.5)	1.765	2.405	3.165
Exponential (B=C=1; G=H=2.718)	1.349	3.005	3.308

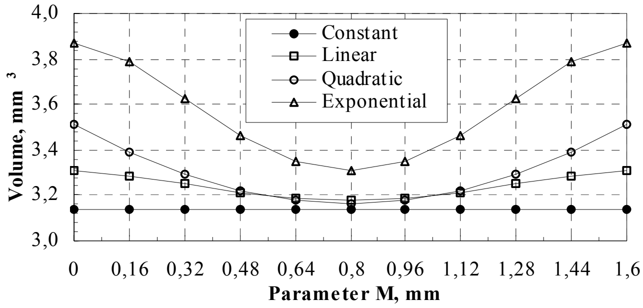

Figure 10. Variation of the cross-section for different geometries.

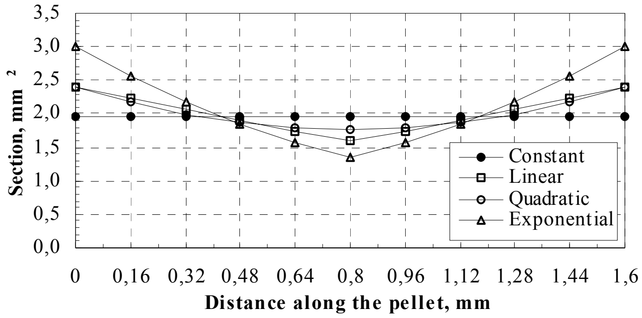

Figure 11. Variation of the volume of the pellet with M for different geometries

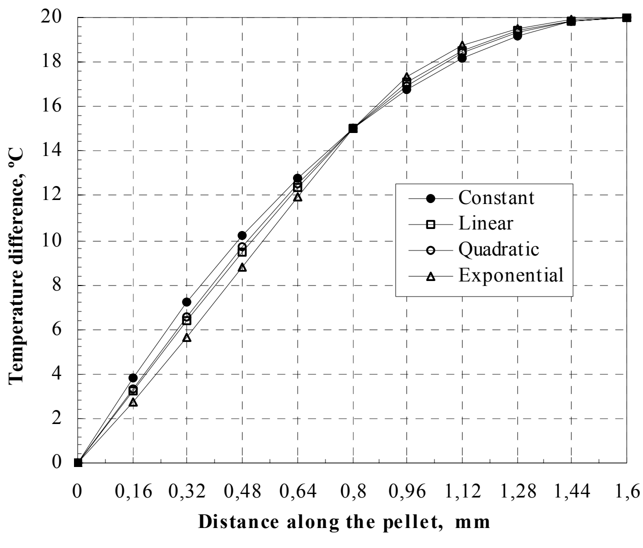

Figure 12. Temperature difference, (T_h-T_c), T_c= 273 K, $I_{\dot{Q}_c}\big|_{max}$

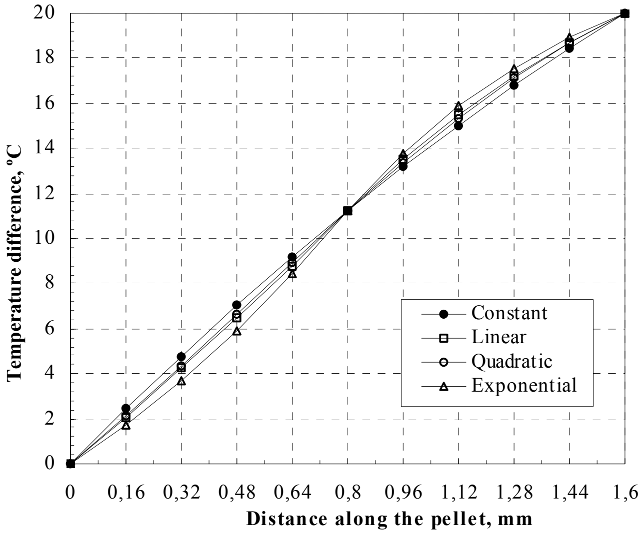

Figure 13. Temperature difference (T_h-T_c), T_c= 273 K, $I_{\phi}\big|_{max}$

The following conclusions were extracted:

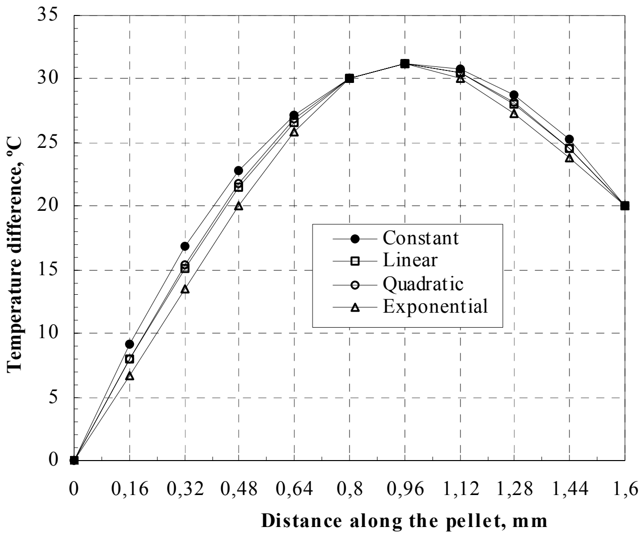

Figure 14. Temperature difference (T_h-T_c), T_c= 273 K, I^*

- The sections at the ends of the pellet increase their values in 21,78 %, 22,65 %, and 53,24 % respect to the constant cross section pellet.
- The total volume of the pellet also is increased in 1,38%, 0,89%, and 5,46%.
- The temperature differences achieves maximum increments of 1,64%, 1,15% and 3,12% when $I_{\dot{Q}_c}\big|_{max}$ is applied and 2,99 %, 2,16 %, and 5,78 % for $I_{\phi}\big|_{max}$.
- In all cases, the thermal powers interchanged and the cooling efficiency have the same values.

Conclusions

This study allows for showing the enormous possibilities of using a pellet geometry distinct to the constant cross section. For example, the use of the exponential variation cross section allows for an increment of the areas at the ends of the pellet of 53,24 % using just only a 5,46 % more thermoelectric material that with the constant cross section pellet. This area increment at the ends of the pellet can favour the heat transmission to the ceramic plates.

It was not an objective of this work to find an optimum geometry because in order to achieve this aim it is necessary to include the rest of elements of the thermoelectric module and the heat dissipation systems. Furthermore, the methodology used in this work has some restrictions to take into account in the analysis of the results obtained.

The thermal and electrical flows were considered one-dimensional, that is, with only x component inside the thermoelement. This is only exact with the constant cross-section. In the rest of geometries, where $da(x)/dx \neq 0$, there are y and z components of the electrical density, and the thermal flow which can obtained variations in the results obtained.

At the same time, the existence of electrical bridges at the ends of the pellet will produced in any cross-section, and particularly in the closest to the ends, modifications in the electrical current and temperature distributions. These variations will modify the fact that changes in the geometry of the pellet do not influence on the values of the interchanged thermal powers and the efficiencies associated, if the total E remains constant.

To solve the restrictions mentioned above, it would be necessary to use three-dimensional, or at least two-dimensional models. The use of numerical methods, such as finite element techniques would allow for evaluating not only the accuracy of the one-dimensional method explained in this work, but also the gain, which from the point of view of specific applications, can cause an increment of the areas at the ends of the pellet.

References

1. Chapman, A.J. "Heat Transfer", 4th ed. Prentice Hall, (New York, 1984), pp. 608.
2. Thacher, E.F. "Shapes which Maximise Thermoelectric Generator Efficiency". *Proc. 4th International Conference on Thermoelectric Energy Conversion*. 1982, Arlington, Texas, USA, pp. 67-74.