upcomillas.es

upcomillas.es

Multicriteria Decision Making

Andrés Ramos (Andres.Ramos@comillas.edu)
Pedro Sánchez (Pedro.Sanchez@comillas.edu)
Sonja Wogrin (Sonja.Wogrin@comillas.edu)

Contents

1

Basic concepts

Decision theory

- Decision: choosing the best of possible $\rightarrow$ To state best and possible
- Possible solutions or feasible decisions:
  - Finite set: alternatives can be enumerated
  - Continuous set: alternatives are defined through constraints
- The Best:
  - One criterion (classic Optimization and classic Decision Theory)
  - Multiple criteria o multiple decision makers (Game Theory and Multicriteria Decision Making)

Which is the best animal of the nature in running, flying and swimming simultaneously?

- The fastest runner?

Cheetah is the fastest running animal in the world

- The fastest flying?

Peregrine falcon is the fastest bird in the world

- The fastest swimmer?

Sailfish is the fastest fish in the world

And the winner is … the DUCK

- Is able to run although less than the cheetah

- Is able to fly although less than the peregrine falcon

- Is able to swim although less than the sailfish

Methods

- Continuous decisions
  - Multiobjective optimization
  - Compromise programming
  - Satisfying methods (goals)
- Discrete decisions
  - Analytical Hierarchic Process (AHP)
  - Outranking methods (Electre, Promethee, ...)
- General statement:

$$\text{opt} \quad z = (z_1(x), \ldots, z_p(x))$$
$$x \in F$$

- F : feasible decision space (continuous: feasible region, $F \subseteq \mathbb{R}^n$)
- $z(F)$: criterion space (numerical criteria: $z(F) \subseteq \mathbb{R}^p$)

Basic concepts

- Attribute: observable "value" (measurable) of an alternative, independent of the decision maker
- Objective: direction to improve an attribute (max. o min. if numerical; otherwise, preferences)
- Target: an acceptable level of achievement for an attribute
- Goal: combination of an attribute with its target
- Criterion: relevant attributes, objectives or goals to a decision problem

Solution concept

Efficiency or Pareto optimality criterion

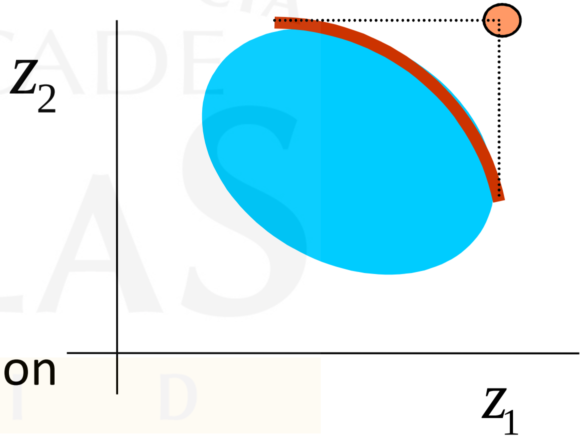
One feasible solution is efficient or Pareto optimal if no other feasible solution can yield an improvement in one objective without causing a degradation in at least another objective

Dominated or non efficient alternative: there is another one with better attributes

Criteria space $\max (z_1, z_2)$

Efficient set or Pareto frontier

Dominated alternatives



Numerical attributes, objectives maximisation

→ Efficient set:

$$\varepsilon = \left\{ x \in F : \nexists x' \in F \text{ with } z_k(x') \geq z_k(x) \ \forall k \text{ and } \exists t \in \{1,\ldots,p\} \text{ with } z_t(x') > z_t(x) \right\}$$

Best compromise solution: efficient solution chosen by the decision maker

2

Continuous methods

Multiobjective optimization

$$\max \quad z = (z_1(x), \ldots, z_p(x))$$
$$x \in F$$

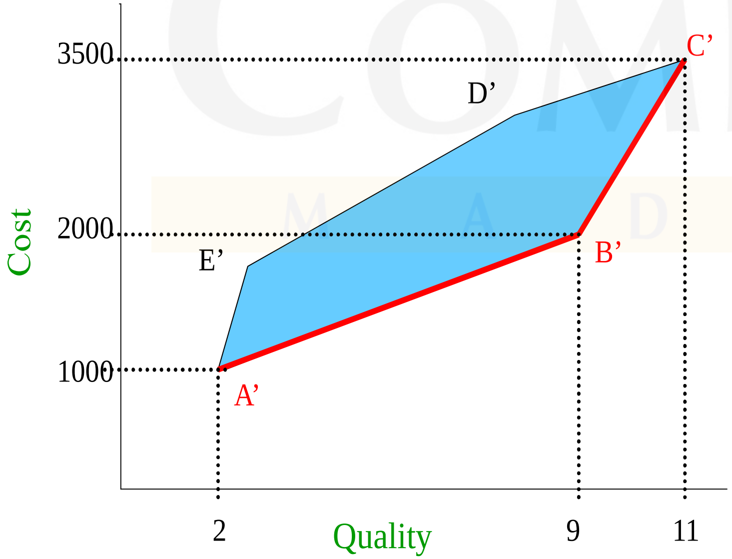
$z_i(x)$: mathematical expression for attribute i

x : decisional variables vector

F : constraints defining feasible solutions

Pay-off matrix: optimum value for an objective (without the others), and values of the other objectives for this solution. Conflict level

Trade-offs: degradation of one objective to improve another one in one unit ("cost" of one objective in terms of another one). Efficient set slopes



Pay-off matrix:

	Cost	Quality
Cost	**1000**	2
Quality	3500	**11**

Trade-offs:

$$T_{A'B'} = \frac{2000-1000}{9-2} = 142.86$$

$$T_{B'C'} = \frac{3500-2000}{11-9} = 750$$

Multiobjective optimization

Weighted-Sum Method (Zadeh, 1963)

Multiplying any objective function with a weight or non negative factor and adding in a single composite objective function

Parametric programming (changing weights the efficient set is obtained)

$$\left.\begin{array}{c} \max \sum_{i=1}^{p} \lambda_i z_i(x) \\ x \in F \\ \lambda \geq 0 \end{array}\right\} P(\lambda)$$

Theorem: If $\lambda_i > 0 \ \forall i$ then any optimal solution of $P(\lambda)$ is efficient.

Converse of theorem is true under some assumptions (convexity, linearity)

Normalized criteria (units)

Multiobjective optimization

Epsilon-Constraint Method

Optimisation of one objective including the rest of objectives as parametric constraints

Parametric programming (changing right hand side, efficient set)

$$\left.\begin{array}{ll} \max & z_l(x) \\ & x \in F \\ & z_k(x) \geq \varepsilon_k \quad k = 1,\ldots,l-1,l+1,\ldots,p \end{array}\right\} P_l(\varepsilon)$$

Theorem: If there is only one solution, it is efficient.

Theorem: If x^* is efficient, $\forall l \quad \exists \varepsilon_k$ such that x^* is optimum of $P_l(\varepsilon)$

Multicriterion simplex method (Zeleny, 1974)

Only for linear objectives and constraints

Each iteration, check efficiency of solutions (extreme points or vertices)

All the efficient vertices are obtained

Efficient set: linear combination of adjacent vertices

Compromise programming

Ideal point: optimum values of each objective subject to problem constraints

$z^* = (z_1^*, \dots, z_i^*, \dots, z_p^*)$ $\quad z_i^* = \max_{x \in F} z_i(x)$ $\quad z_i^*$: anchor value

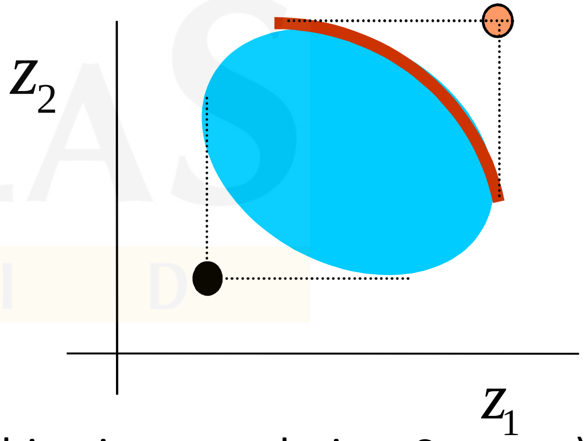
Optimum element or best-compromise solution: efficient solution closest to the ideal point (Zeleny's axiom of choice, 1976)

Degree of closeness between attribute i-th and its anchor value, normalized

z_{*i} anti-ideal or nadir point (worst value criteria in efficient set)

$$d_i(x) = \frac{z_i^* - z_i(x)}{z_i^* - z_{*i}}$$

$$\min_{x \in F} L_\pi = \left[\sum_{i=1}^{p} w_i^\pi \left(\frac{z_i^* - z_i(x)}{z_i^* - z_{*i}} \right)^\pi \right]^{1/\pi}$$



Weights, importance of discrepancy of criteria (subjective → ordering, Saaty, ...)

π=∞ → Tchebychev or minimising maximum distance (linear)

π=1 → lineal addition of weighted distances (linear)

Compromise set: varying π (usually $[L_1, L_\infty]$)

Goal programming

Satisfying logic (Simon, 1955):

decision-making behaviour where, instead of attempting to optimize system performance, a level of aspiration is set either subjectively or heuristically and no further effort is expended to exceed that level of performance → Goal programming (Charnes y Cooper(61), Lee (72) e Ignizio (76))

Attribute → mathematical expression: $z_i(x)$

Target or level of aspiration: acceptable level of achievement $\hat{z}_i$

Goal: $z_i(x) \geq \hat{z}_i$

With deviation variables: $z_i(x) + n_i - p_i = \hat{z}_i$

Deviation variables to be minimized:

if goal is "at least" one value, n_i ; if it is "at most", p_i

Model (if at least): $$\min_{x \in F \cap \text{goal constraints}} \sum_{i=1}^{p} n_i$$

Goal programming: variations

- *Weighted Goal Programming:*

$$\min \sum_{i=1}^{p} (\alpha_i n_i + \beta_i p_i)$$

$$z_i(x) + n_i - p_i = \hat{z}_i \quad i = 1,\dots,p$$

$$x \in F$$

$$n_i \geq 0,\ p_i \geq 0 \quad i = 1,\dots,p$$

(dividing by the target: %)

- *MINIMAX or Tchebychev Goal Programming:* (balanced solution)

$$\min \quad D$$

$$\alpha_i n_i + \beta_i p_i \leq D \quad i = 1,\dots,p$$

$$z_i(x) + n_i - p_i = \hat{z}_i \quad i = 1,\dots,p$$

$$x \in F$$

$$n_i \geq 0,\ p_i \geq 0 \quad i = 1,\dots,p$$

- *Lexicographic Goal Programming:* Priority levels goals (pre-emptive). To solve: sequentially each level, keeping values of unwanted deviation variables previously achieved

$$Lex \min a = [g_1(n,p) + g_2(n,p), g_3(n,p), g_4(n,p) + g_5(n,p) + g_6(n,p)]$$

Andrés Ramos

http://www.iit.upcomillas.es/aramos/

Andres.Ramos@comillas.edu

Pedro Sánchez

Pedro.Sanchez@comillas.edu

Sonja Wogrin

Sonja.Wogrin@comillas.edu

Departamento de Organización Industrial
Alberto Aguilera, 23
28015 Madrid, Spain
Tel +34 91 542 28 00
Fax + 34 91 542 11 32
info-doi@doi.icai.upcomillas.es