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Linear Programming

Andrés Ramos (Andres.Ramos@comillas.edu)
Pedro Sánchez (Pedro.Sanchez@comillas.edu)
Sonja Wogrin (Sonja.Wogrin@comillas.edu)

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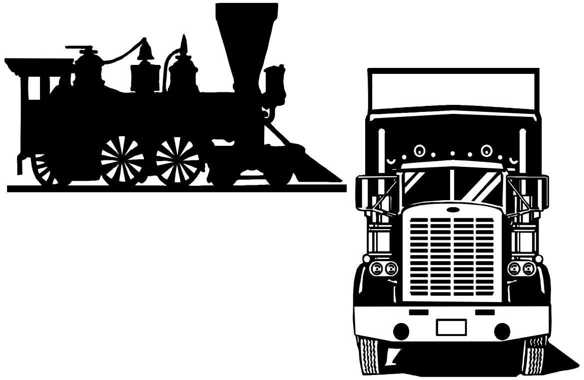
Graphical Solution

Graphical solution(i)



Example:

- A craftsman builds toy trains and trucks
- Using screws, blocks, and tires as components
- Per week, he disposes of 8000, 6000, and 6300 of each component, respectively
- The profits per train and truck are 1.6 euros/unit and 1.4 euros/unit

	Screws	Blocks	Tires
Train	10	15	18
Truck	20	10	6

Graphic solution (ii)

Mathematical Formulation:

$$\max z = 1.6x_1 + 1.4x_2$$ → Maximization of total profits

$$subject\ to:$$

$$10x_1 + 20x_2 \leq 8000$$ → Limitation of SCREWS

$$15x_1 + 10x_2 \leq 6000$$ → Limitation of BLOCKS

$$18x_1 + 6x_2 \leq 6300$$ → Limitation of TIRES

$$x_1, x_2 \geq 0$$

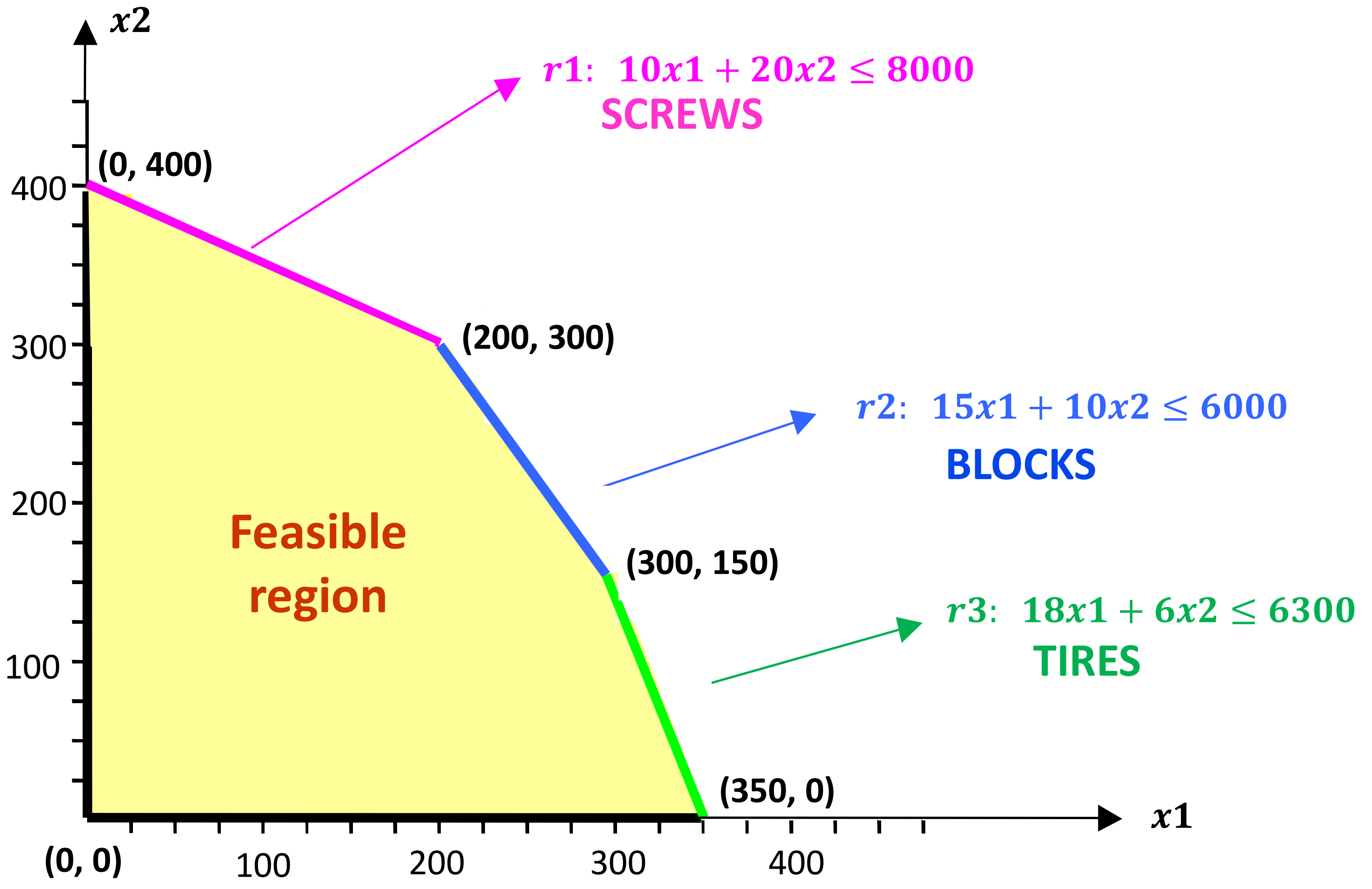
x_1 : **Number of produced trains**

x_2 : **Number of produced trucks**

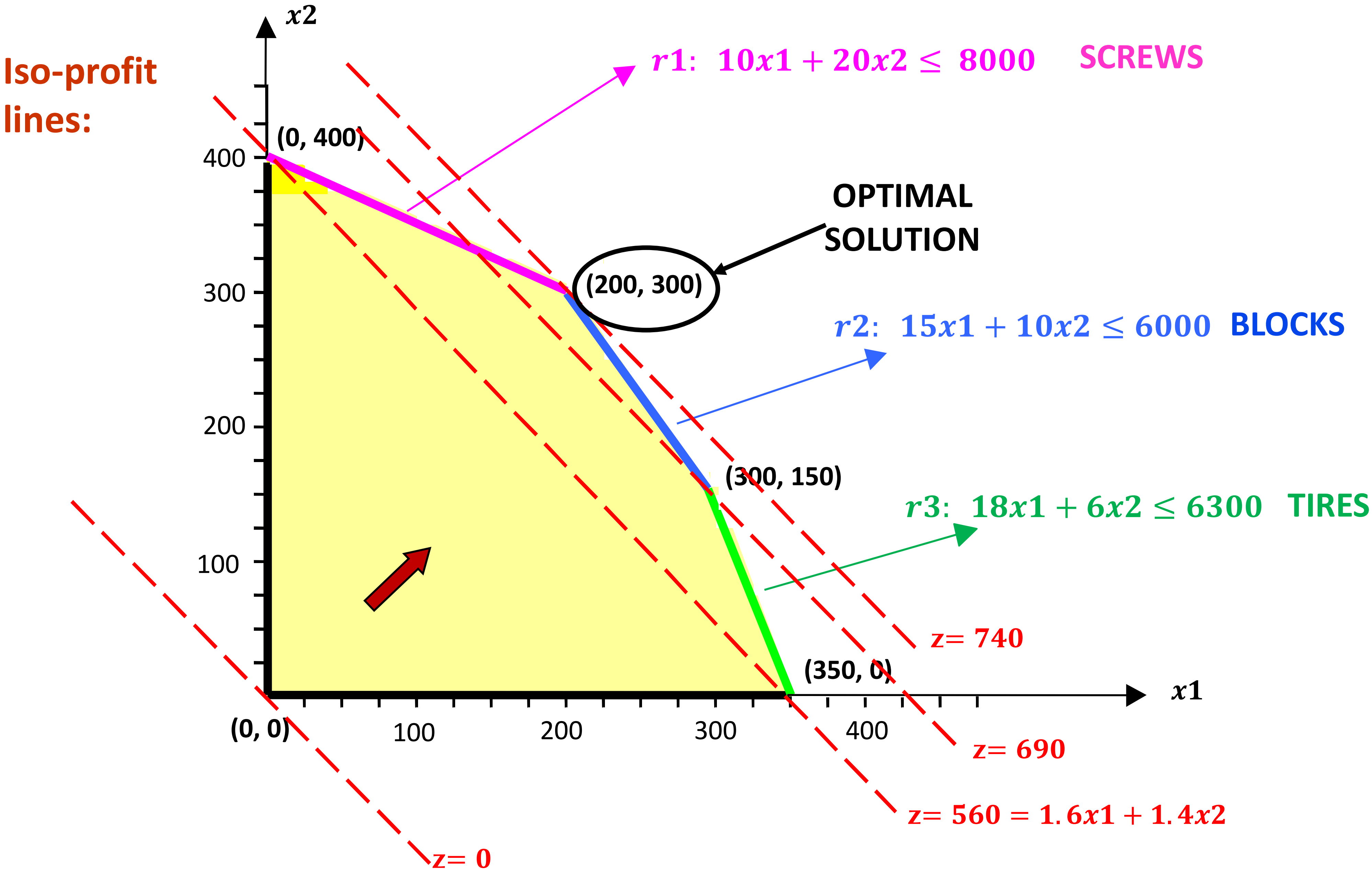
The variables are considered as continuous as a first approximation

Graphical solution (iii)

Graphical representation of constraints



Graphical solution (iv)



Graphical solution (v)

Optimal solution:

- The optimal solution is to produce **200 trains** and **300 trucks**
- All the **screw** components are used (**8000**)
- All the **block** components are used (**6000**)
- Only **5400 tires** are used. **900 tires** are left.
- The resulting total profit is **740 euros**

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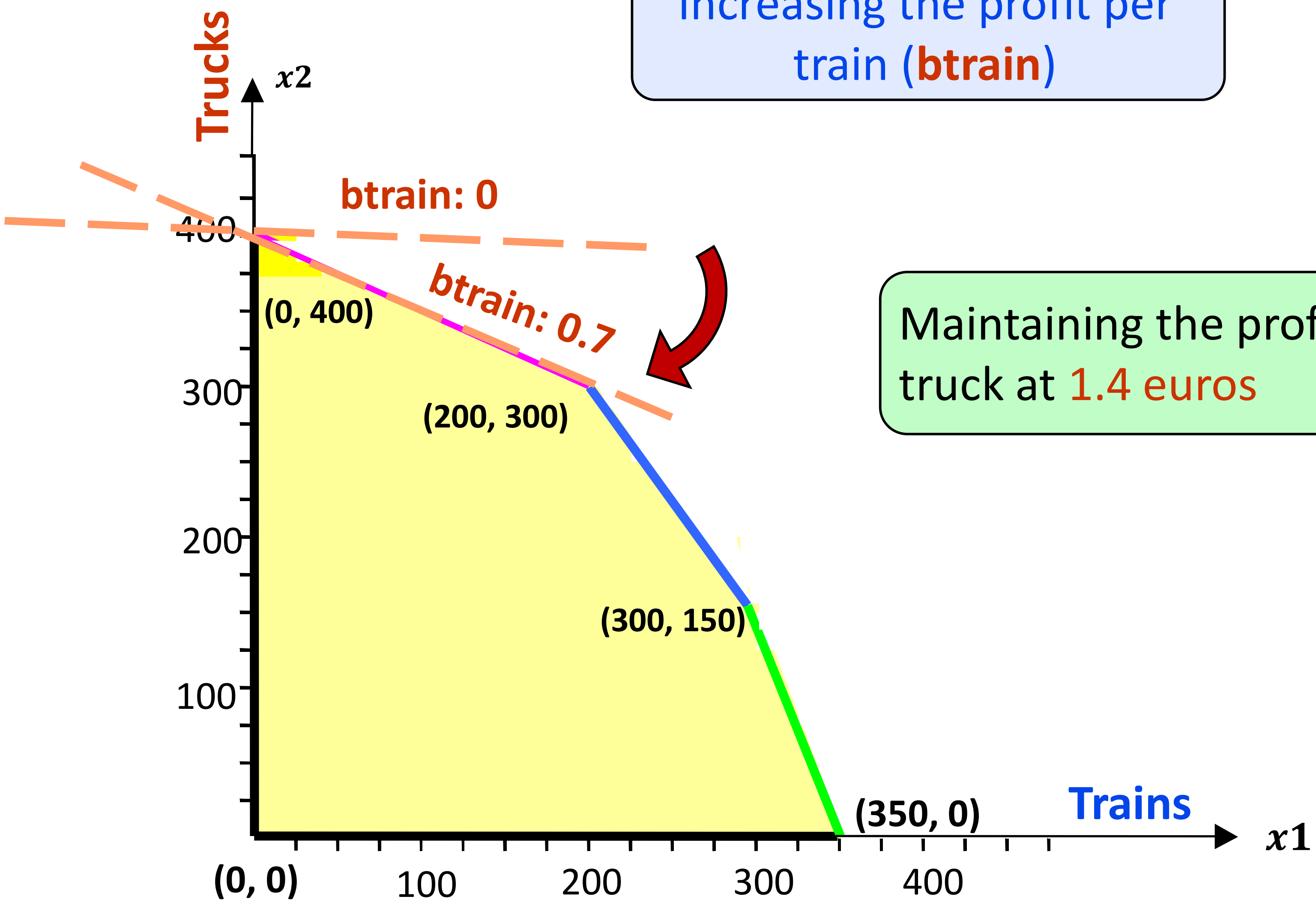
Graphical Sensitivities

Graphical sensitivities (i)

Changes of the objective function coefficients (i)

Increasing the profit per train (**btrain**)

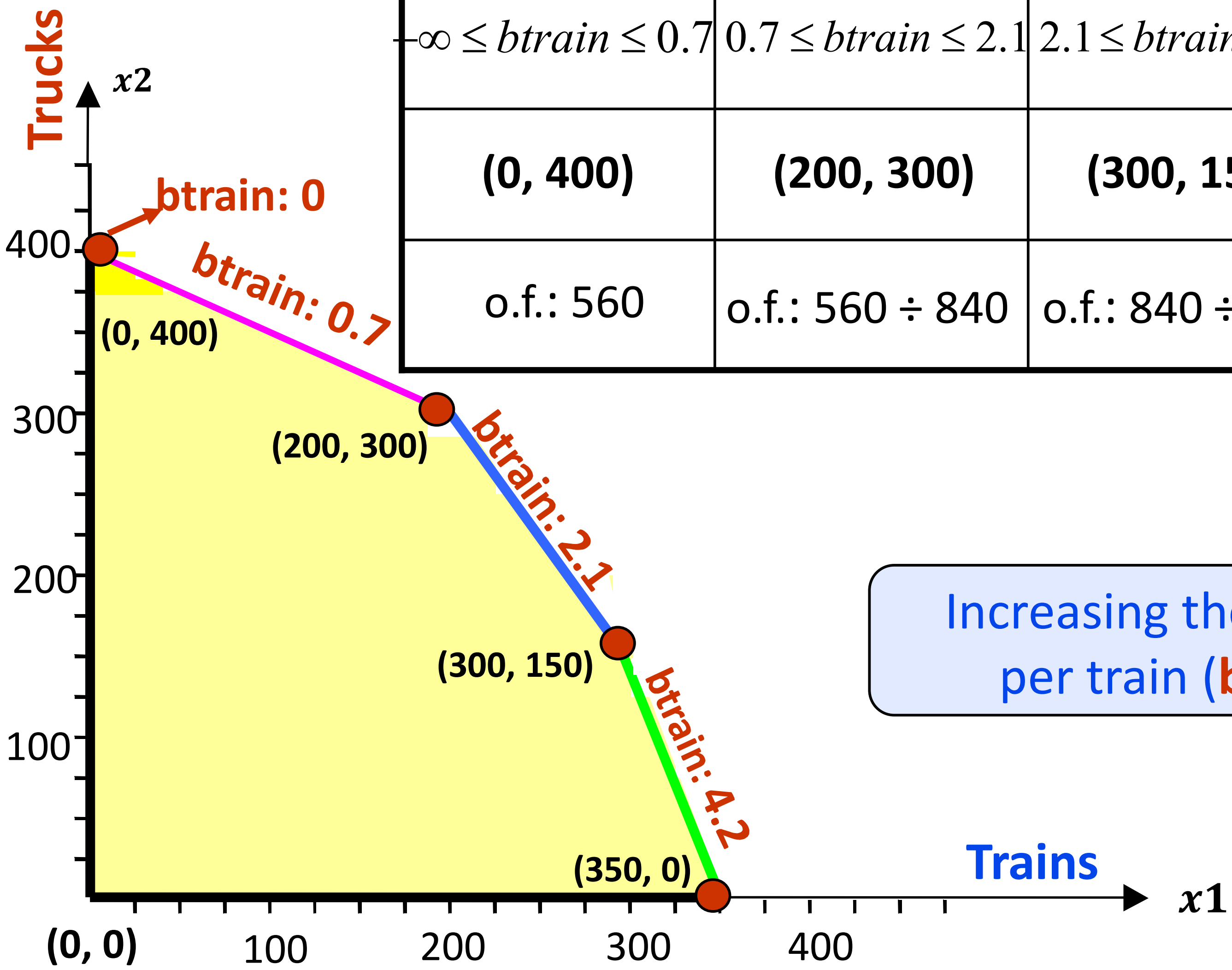


Maintaining the profit per truck at 1.4 euros

Graphical sensitivities (ii)

Changes of the objective function coefficients (ii)

$-\infty \le btrain \le 0.7$	$0.7 \le btrain \le 2.1$	$2.1 \le btrain \le 4.2$	$4.2 \le btrain \le +\infty$
(0, 400)	**(200, 300)**	**(300, 150)**	**(350, 0)**
o.f.: 560	o.f.: 560 ÷ 840	o.f.: 840 ÷ 1470	o.f.: 1470 ÷ ∞

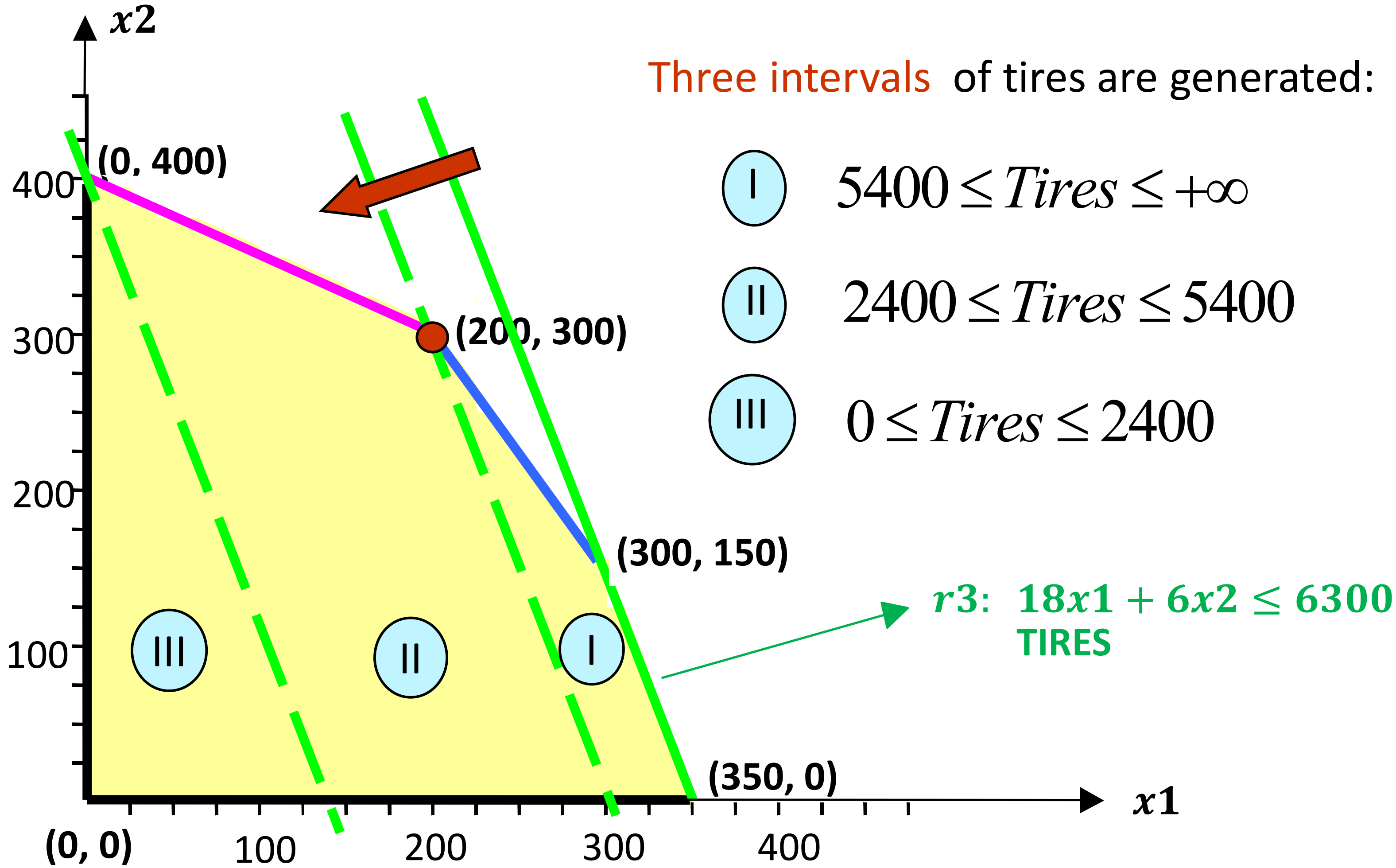


Increasing the profits per train (**btrain**)

Graphical sensitivities (iii)

Changes in the constraints (i)

Reducing the number of existing tire components and maintaining the objective function



Three intervals of tires are generated:

I $5400 \leq Tires \leq +\infty$

II $2400 \leq Tires \leq 5400$

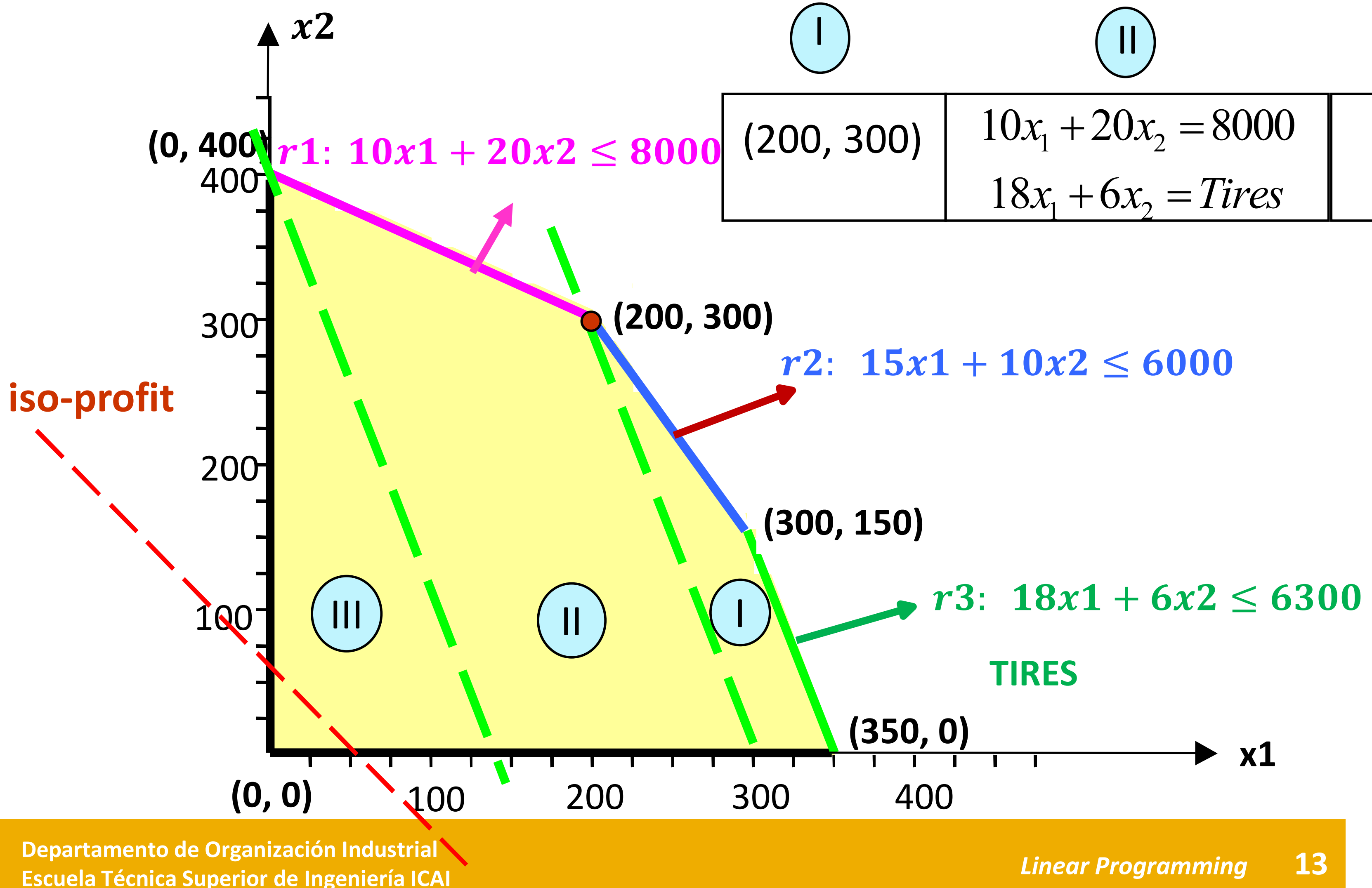
III $0 \leq Tires \leq 2400$

Graphical sensitivities (iv)

Changes in the constraints (ii)

The **solution** in each of the intervals is:

I	II	III
(200, 300)	$10x_1 + 20x_2 = 8000$ $18x_1 + 6x_2 = Tires$	$x_1 = 0$ $18x_1 + 6x_2 = Tires$



Graphical sensitivities (v)

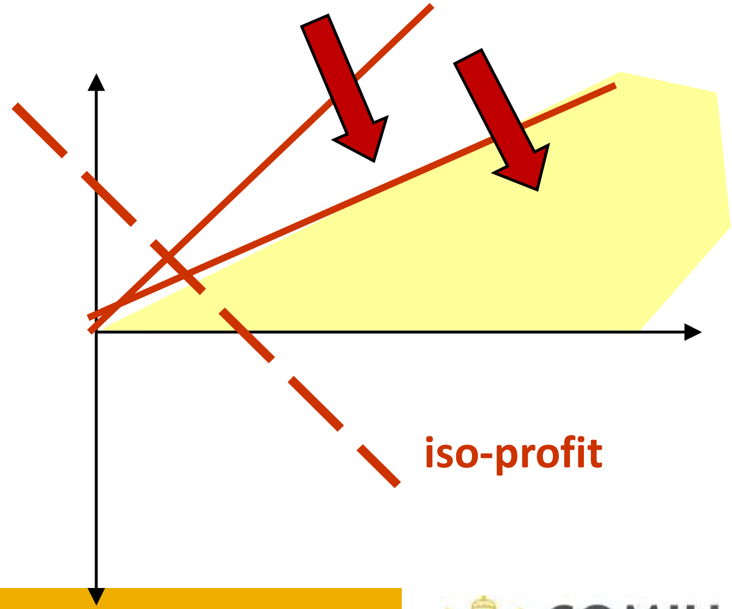
Unbounded solution

A **bounded solution** to an optimization problem does not necessarily always exist.

In the example of the craftsman, we obtain an unbounded solution if:

- We don't limit the existing components of screws, blocks, and tires
- If the component provider mandates us to use at least 4 blocks and 4 tires for each 4 screws

$$(10x_1 + 20x_2)/4 \leq (15x_1 + 10x_2)/4$$
$$(10x_1 + 20x_2)/4 \leq (18x_1 + 6x_2)/4$$



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Simplex Method

Simplex Method (i)

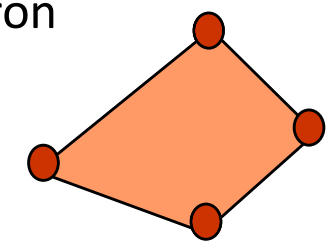
Geometry of linear programming

- Polyhedron: Region defined as the intersection of a finite set of hyperplanes

Half-spaces
$$\begin{cases} \sum_j a_{ij} x_j \le b_i \\ \sum_j a_{ij} x_j \ge b_i \end{cases}$$

- The feasible region of a linear programming problem is a polyhedron

- Vertex of a polyhedron: the point that cannot be expressed as a convex linear combination of two different points of the polyhedron



If an optimal solution of a linear programming problem exists, then it can be found in a VERTEX

Simplex Method (ii)

- **Standard form**

$$\min\ z = c^T x$$
$$Ax = b$$
$$x \geq 0$$
$$x \in R^n;\ A \in R^{mxn};\ c \in R^n; b \in R^m$$

Craftsman problem

$$\begin{aligned}
&\min z = -1.6x_1 - 1.4x_2 + 0h_1 + 0h_2 + 0h_3 \\
&\textit{subject to:}: \\
&10x_1 + 20x_2 + h_1 = 8000 \\
&15x_1 + 10x_2 + h_2 = 6000 \\
&18x_1 + 6x_2 + h_3 = 6300 \\
&x_1, x_2, h_1, h_2, h_3 \geq 0
\end{aligned}$$

The simplex method usually requires iterations proportional to the number of constraints (m is the number of constraints), and the solution time is proportional to m^3

$$c^T = (-1.6, -1.4, 0, 0, 0)$$
$$x^T = (x_1, x_2, h_1, h_2, h_3)$$
$$A = \begin{bmatrix} 10 & 20 & 1 & 0 & 0 \\ 15 & 10 & 0 & 1 & 0 \\ 18 & 6 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 8000 \\ 6000 \\ 6300 \end{bmatrix}$$

Simplex Method (iii)

Transformations into Standard Form

- **Objective Function:** $\max z \rightarrow \min -z$

- **Constraints ≤:** A slack variable u_i is introduced

$$\sum_j a_{ij} x_j \le b_i \rightarrow \sum_j a_{ij} x_j + u_i = b_i \qquad \boxed{b_i \ge 0}$$

- **Constraints ≥:** An excess variable v_i is introduced

$$\sum_j a_{ij} x_j \ge b_i \rightarrow \sum_j a_{ij} x_j - v_i = b_i \qquad \boxed{b_i \ge 0}$$

- **Negative variables:** $-\infty \le x_j \le 0 \Rightarrow x_j = -y_j ;\; 0 \le y_j \le +\infty$

- **Negative bounded variables :** $L_j \le x_j \le 0 \Rightarrow x_j = y_j + L_j ;\; 0 \le y_j \le -L_j$

- **Free variables:** Substitution for two positive vars. $x_j = x_j^+ - x_j^-$

$$0 \le x_j^+ \le \infty \qquad 0 \le x_j^- \le \infty$$

Simplex Method (iv)

Types of Solutions

- **Feasible Solution:** Satisfies all the constraints
  Belongs to the feasible region

- **Infeasible Solution:** Violates at least one constraint
  Does not belong to the feasible region

- **Basic Feasible Solution:** Has m basic variables associated to the non-singular constraint matrix A of rank m and can take a value ≠0, while the rest of the variables (non-basic variables) = 0

- **Optimal Solution:** Basic feasible solution with best o.f. value

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Simplex Algorithm

Simplex Algorithm (i)

$$\min\ z = c^T x$$
$$Ax = b$$
$$x \geq 0$$
$$x \in R^n;\ A \in R^{mxn};\ c \in R^n; b \in R^m$$

Partition into BASIC and NON-BASIC variables

$$x^T = [x_B, x_N]^T \quad \begin{cases} x_B \in R^m & \text{vector of BASIC variables} \\ x_N \in R^{n-m} & \text{vector of NON-BASIC variables} \end{cases}$$

$$A = [B, N]^T \quad \begin{cases} B \in R^{m \times m} & \text{BASIC matrix} \\ N \in R^{m \times n-m} & \text{NON-BASIC matrix} \end{cases}$$

$$c^T = [c_B, c_N]^T \quad \begin{cases} c_B \in R^m & \text{coefficients of BASIC variables} \\ c_N \in R^{n-m} & \text{coefficients of NON-BASIC variables} \end{cases}$$

Simplex Algorithm (ii)

Reformulating the problem

$$Ax = b$$

$$Bx_B + Nx_N = b \quad \rightarrow \; x_B = B^{-1}\left(b - N x_N\right) = B^{-1} b - B^{-1} N x_N$$

Objective function

$$z = c_B^T x_B + c_N^T x_N = c_B^T B^{-1} b + \left[c_N^T - c_B^T B^{-1} N \right] x_N$$

Since the non-basic variables take value zero, we obtain:

$$x_B = B^{-1} b$$

$$z = c_B^T x_B = c_B^T B^{-1} b$$

Simplex Algorithm (iii)

The reduced costs

$$\hat{c}_N^{\ T} = c_N^{\ T} - c_B^{\ T} B^{-1} N$$ → Vector of reduced costs

$$\hat{c}_j = c_j - c_B^{\ T} B^{-1} a_j = c_j - z_j$$

a_j: column of variable x_j in A

The reduced costs in the objective function:

$$z = c_B^{\ T} x_B + \sum_{j \in I_N} \hat{c}_j x_j = \hat{z} + \sum_{j \in I_N} \hat{c}_j x_j = \hat{z} + \sum_{j \in I_N} \left(c_j - z_j\right) x_j$$

- The reduced costs indicate the <u>change of the o.f.</u> due to a unitary increment of the variable
- At the optimum, all the reduced costs of the non-basic variables are ≥ 0 (standard)
- The reduced costs <u>of basic variables</u> are <u>always</u> 0
- If the reduced cost of a non-basic variable <u>at the optimum is</u> 0 → <u>multiple optima</u>

Simplex Algorithm (iv)

Algorithm applied to craftsman problem:

$$x_B = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad c_N = \begin{pmatrix} -1.6 \\ -1.4 \end{pmatrix} \quad c_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 10 & 20 \\ 15 & 10 \\ 18 & 6 \end{pmatrix}$$

$$x_B = B^{-1} b = \begin{pmatrix} 8000 \\ 6000 \\ 6300 \end{pmatrix} \quad x_N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

→ Initial solution

$$(x_1, x_2) = (0, 0)$$

Reduced costs: $\hat{c}_N{}^T = c_N{}^T - c_B{}^T B^{-1} N = \begin{pmatrix} -1.6 & -1.4 \end{pmatrix}$

Changes in basic variables:

$$x_B = B^{-1} b - B^{-1} N x_N \qquad x_B = \begin{pmatrix} 8000 \\ 6000 \\ 6300 \end{pmatrix} - \begin{bmatrix} 10 & 20 \\ 15 & 10 \\ 18 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Simplex Algorithm (v)

Evolution of the o.f. in the problem of the craftsman:

The o.f. changes its value when there is a change of the non-basic variables

Example: $$z = c_B{}^T B^{-1} b + \hat{c}_N{}^T x_N = 0 + (-1.6, -1.4) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1.6x_1 - 1.4x_2$$

The o.f. decreases when increasing any of the non-basic variables → The actual basis is NOT optimal

- The optimality check is to verify if there is a variable with negative reduced cost

  ① If there is not, then the current basis is optimal

  ② If there exists at least one, then choose (among the negative ones) the entering basic variable EBV (the one with the reduced cost with the highest absolute value)

Example: var x_1 has a reduced cost of –1.6 and hence is the **entering basic variable**

Simplex Algorithm (vi)

Increasing the NON-BASIC variable:

The NON-BASIC variable can be increased until the BASIC variables violate at least one of the constraints

$$x_B = B^{-1}b - B^{-1}N\,x_N = \hat{b} - B^{-1}a_t\,x_t = \hat{b} - y_t\,x_t$$

a_t: column of matrix A corresponding to the variable x_t

$$(x_B)_i = \hat{b}_i - y_{it}\,x_t \quad ; \quad i \in I_N$$

- If $y_{it} > 0 \rightarrow (x_B)_i$ decreases when x_t increases until $\frac{\hat{b}_i}{y_{it}}$ (until it is zero)

- If $y_{it} \leq 0 \rightarrow (x_B)_i$ increases or remains the same

Simplex Algorithm (vii)

Maximum increase of the NON-BASIC variable:

The NON-BASIC variable chosen as **ENTERING BASIC VARIABLE (EBV)** is increased until the first BASIC variable goes to zero

The **LEAVING BASIC VARIABLE (LBV)** is the first one to go to zero

$$\bar{x}_t = \min_{1 \le i \le m} \left\{ \frac{\hat{b}_i}{y_{it}} : y_{it} > 0 \right\}$$

Example of craftsman:

$$y_1 = B^{-1} a_{x1} = \begin{bmatrix} 10 \\ 15 \\ 18 \end{bmatrix} \quad \bar{x}_1 = \min \left\{ \frac{8000}{10}, \frac{6000}{15}, \frac{6300}{18}, \right\} = \min \{800, 400, \underline{\underline{350}}\}$$

h_3

h_3 is the **leaving basic variable**

Simplex Algorithm (viii)

Updating the BASIC variables and the o.f.:

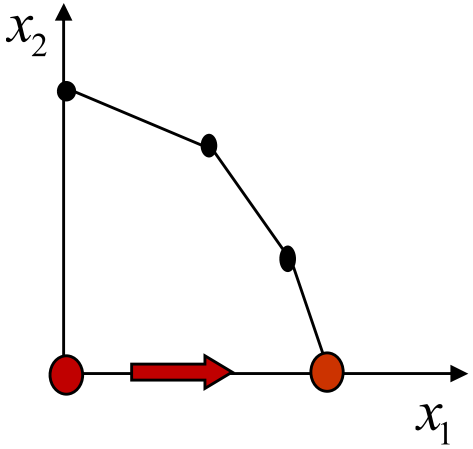
$$x_B = B^{-1} b$$

$$\hat{Z} = c_B^{\ T} x_B$$

Example:

$$B = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 15 \\ 0 & 0 & 18 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 & -0.\widehat{5} \\ 0 & 1 & -0.8\widehat{3} \\ 0 & 0 & 0.0\widehat{5} \end{bmatrix}$$

$$x_B = \begin{bmatrix} h_1 \\ h_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.\widehat{5} \\ 0 & 1 & -0.8\widehat{3} \\ 0 & 0 & 0.0\widehat{5} \end{bmatrix} \begin{bmatrix} 8000 \\ 6000 \\ 6300 \end{bmatrix} = \begin{bmatrix} 4500 \\ 750 \\ 350 \end{bmatrix}$$



Solution 1st iteration: (350, 0) → Objective function= -560

Simplex Algorithm (ix)

STEPS of the SIMPLEX method (i)

① **Initialization**: Initial basic feasible solution

$$x_B = \hat{b} = B^{-1} b \geq 0$$

$$\hat{z} = c_B^{\ T} x_B$$

② **Optimality check:** Calculate the reduced costs

$$\hat{c}_N^{\ T} = c_N^{\ T} - c_B^{\ T} B^{-1} N$$

- If all the reduced costs are ≥ 0, the basis is optimal (when the problem is a maximization problem, then for optimality: reduced costs ≤ 0)

- If there exists at least one reduced cost < 0, then select this variable as entering basic variable x_t, the negative one with the highest absolute value

Simplex Algorithm (x)

STEPS of the SIMPLEX method (ii)

③ Iteration

$$y_t = B^{-1} a_t$$: pivot column of x_t

$$\frac{\hat{b}_s}{y_{st}} = \min_{1 \le i \le m} \left\{ \frac{\hat{b}_i}{y_{it}} : y_{it} > 0 \right\}$$: leaving basic variable x_s

If all the $y_{it} \le 0$ the problem is unbounded

④ Pivoting

- Update matrix B^{-1}
- Update the vector of basic variables x_B
- Go to step ②

Simplex Algorithm (xi)

Simplex steps for the craftsman problem:

Iteration 2

$$x_B = \begin{pmatrix} h_1 \\ h_2 \\ x_1 \end{pmatrix} \quad x_N = \begin{pmatrix} h_3 \\ x_2 \end{pmatrix} \qquad x_B = \begin{pmatrix} 4500 \\ 750 \\ 350 \end{pmatrix}$$

Reduced costs:

$$\hat{c}_N{}^T = c_N{}^T - c_B{}^T B^{-1} N = (0,-1.4) - (0,0,-1.6) \begin{pmatrix} 1 & 0 & -0.\widehat{5} \\ 0 & 1 & -0.8\widehat{3} \\ 0 & 0 & 0.0\widehat{5} \end{pmatrix} \begin{pmatrix} 0 & 20 \\ 0 & 10 \\ 1 & 6 \end{pmatrix} = \left(0.0\widehat{8}, \underline{\underline{-0.8\widehat{6}}}\right)$$

Select variable x_2 as entering basic variable

$$y_t = B^{-1} a_{x_2} = \begin{pmatrix} 16.\widehat{6} \\ 5 \\ 0.3 \end{pmatrix} \qquad \bar{x}_t = \min\left\{\frac{4500}{16.\widehat{6}}, \frac{750}{5}, \frac{350}{0.3}\right\} = \min\left\{270, \underline{\underline{150}}, 1166.\widehat{6}\right\}$$

Select variable h_2 as leaving basic variable

Simplex Algorithm (xii)

Simplex steps for the craftsman problem:

Iteration 2 (cont.)

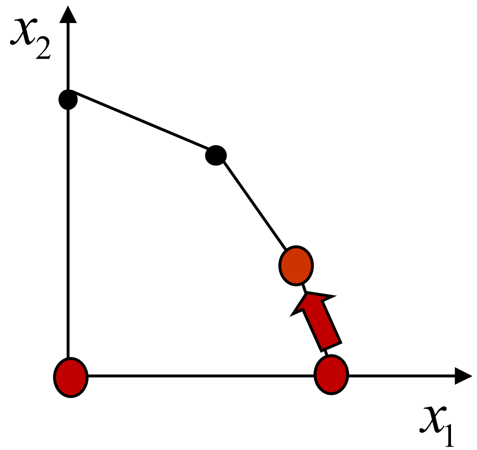
$$x_B = \begin{pmatrix} h_1 \\ x_2 \\ x_1 \end{pmatrix} = B^{-1}\,b = \begin{pmatrix} 1 & -3.\widehat{3} & 2.\widehat{2} \\ 0 & 0.2 & -0.1\widehat{6} \\ 0 & -0.0\widehat{6} & 0.\widehat{1} \end{pmatrix} \begin{pmatrix} 8000 \\ 6000 \\ 6300 \end{pmatrix} = \begin{pmatrix} 2000 \\ 150 \\ 300 \end{pmatrix}$$

Solution 2nd Iteration

$$(x_1, x_2) = (300, 150)$$

Objective function: -690

The solutions are changing from adjacent vertices



The objective function always improves or stays the same

Simplex Algorithm (xiii)

Simplex steps for the craftsman problem:

Iteration 3

$$x_N = \begin{pmatrix} h_3 \\ h_2 \end{pmatrix}$$

Reduced costs:

$$\hat{c}_N^{\,T} = c_N^{\,T} - c_B^{\,T} B^{-1} N = (0,0) - (0,-1.4,-1.6) \begin{pmatrix} 1 & -3.\widehat{3} & 2.\widehat{2} \\ 0 & 0.2 & -0.16 \\ 0 & -0.0\widehat{6} & 0.\widehat{1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \left(\underline{\underline{-0.0\widehat{5}}}, 0.17\widehat{3}\right)$$

Select variable h_3 as entering basic variable

$$y_t = B^{-1} a_{h_3} = \begin{pmatrix} 2.\widehat{2} \\ -0.1\widehat{6} \\ 0.\widehat{1} \end{pmatrix} \qquad \bar{x}_t = \min\left\{ \frac{2000}{2.\widehat{2}}, null, \frac{300}{0.\widehat{1}} \right\} = \min\left\{ \underline{\underline{900}}, null, 2700 \right\}$$

Select variable h_1 as leaving basic variable

Simplex Algorithm (xiv)

Simplex steps for the craftsman problem:

Iteration 3 (cont.)

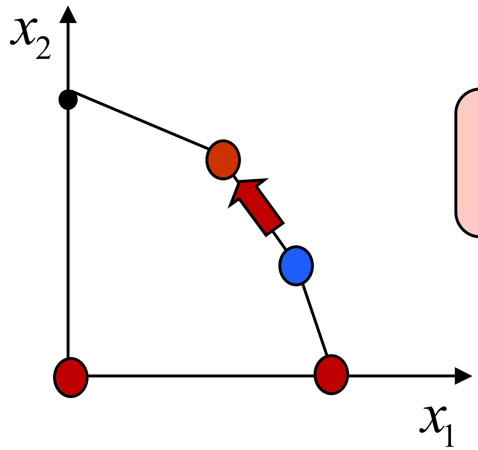
$$x_B = \begin{pmatrix} h_3 \\ x_2 \\ x_1 \end{pmatrix} = B^{-1} b = \begin{pmatrix} 0.45 & -1.5 & 1 \\ 0.075 & -0.05 & 0 \\ -0.05 & 0.1 & 0 \end{pmatrix} \begin{pmatrix} 8000 \\ 6000 \\ 6300 \end{pmatrix} = \begin{pmatrix} 900 \\ 300 \\ 200 \end{pmatrix}$$

→ Solution 3rd Iteration

Objective Function: -740

$$(x_1, x_2) = (200, 300)$$

Graphically we know it is the optimum



Still needs optimality check

↓

Iteration 4

Simplex Algorithm (xv)

Simplex steps for the craftsman problem:

Iteration 4

$$x_B = \begin{pmatrix} h_3 \\ x_2 \\ x_1 \end{pmatrix} \quad x_N = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \qquad x_B = \begin{pmatrix} 900 \\ 300 \\ 200 \end{pmatrix}$$

Reduced costs:

$$\hat{c}_N^{\,T} = c_N^{\,T} - c_B^{\,T} B^{-1} N = (0,0) - (0,-1.4,-1.6) \begin{pmatrix} 0.45 & -1.5 & 1 \\ 0.075 & -0.05 & 0 \\ -0.05 & 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = (0.025, 0.09)$$

There are no negative reduced costs

Therefore, we have reached the OPTIMUM in the previous iteration

Simplex Algorithm (xvi)

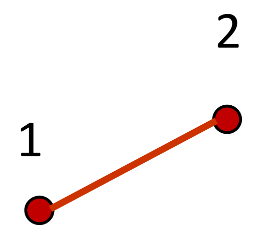
Multiple optima:

You can detect multiple optima when there are non-basic variables whose **reduced costs** are **0**

To be different optima, the non-basic variable with 0 reduced cost needs to take a value ≠0 when entering the basis

Example: Two optimal solutions $\left(x_1^1, x_2^1\right)$ $\left(x_1^2, x_2^2\right)$

$$0 \le \lambda \le 1 \qquad \begin{cases} x_1 = \lambda x_1^1 + (1-\lambda) x_1^2 \\ x_2 = \lambda x_2^1 + (1-\lambda) x_2^2 \end{cases}$$



Simplex Algorithm (xvii)

Initial basic feasible solution

- In constraint ≤ the basis is formed by the slack variable h_i
- In constraint ≥ we introduce an excess variable e_i and an artificial variable a_i
- In constraint = we introduce an artificial variable a_i

$$\sum_j a_{ij} x_j \le b_i \Rightarrow \sum_j a_{ij} x_j + h_i = b_i$$

$$\sum_j a_{ij} x_j \ge b_i \Rightarrow \sum_j a_{ij} x_j - e_i + a_i = b_i$$

$$\sum_j a_{ij} x_j = b_i \Rightarrow \sum_j a_{ij} x_j + a_i = b_i$$

Simplex Algorithm (xviii)

Initial basic feasible solution (cont.)

- The artificial variables (in constraints = or ≥) and the slack variables (in constraints ≤) form the initial basis

- The solutions with artificial variables are not feasible (for the original problem)

- If the artificial variables cannot be eliminated → problem is INFEASIBLE

Simplex Algorithm (xix)

Methods to eliminate artificial variables

- **Big M Method**
  - Introduce artificial variables in the o.f. and penalize them (in the o.f.) with a large coefficient M
  - This method can lead to numerical problems in the resolution

- **Two-phase Method**
  - Phase I uses as o.f. the sum of the artificial variables (and the original constraints). Solve this problem until optimality.

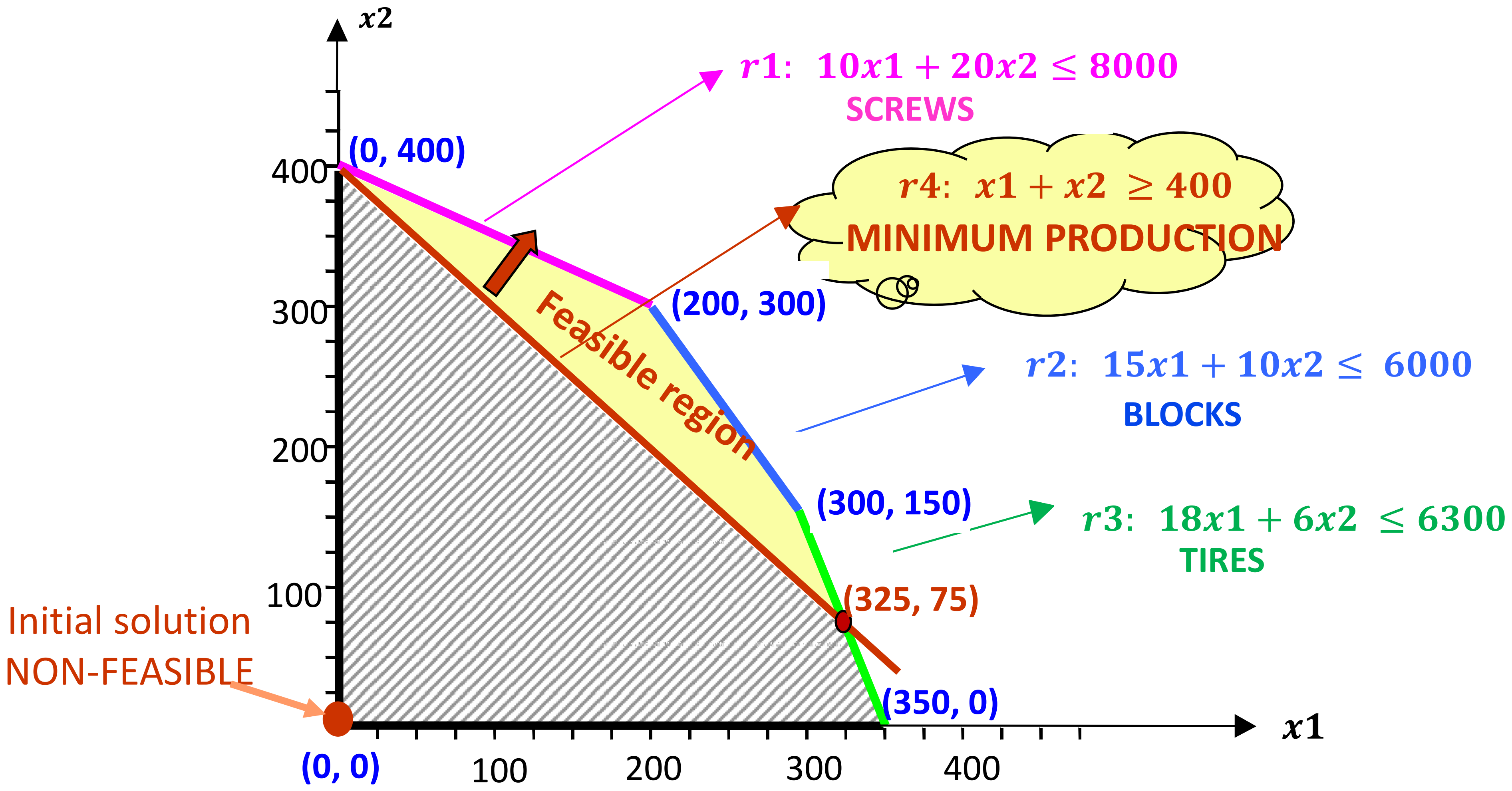
  - If Phase I ends with an o.f. value of zero, then the original problem is feasible, and the current solution is a basic feasible solution of the original problem.

  - Phase II reinstates the original o.f. and starts from the feasible solution obtained at the end of Phase I

Simplex Algorithm (xx)

Example of Two-phase Method

In the example of the craftsman, we introduce the additional constraint that the total number of produced toys needs to be at least 400.



Special cases

- Conflict among entering basic variables
  - Variables with the same reduced cost in the objective function: any one can be selected.
- Conflict among leaving basic variables, DEGENERACY
  - Basic variables that become simultaneously 0 when increasing the entering basic variable. The non leaving basic variables take a value 0 (degenerate variables).
  - If these degenerate basic variables continue to have a value of 0 in another iteration, the entering basic variable cannot increase its value; neither the o.f.
  - If the o.f. cannot increase, a CYCLE can appear (move by a sequence of bases that define the same extreme point). Solvable by special procedures (lexicographic method).
- Non-existence of leaving basic variable, UNBOUNDED SOLUTION
  - Entering basic variable can increase indefinitely with no current basic variable becoming 0. Coefficients of the pivot column are negative or zero ($y_{it} \leq 0$).

How to identify infeasible or unbounded problems?

- Infeasible problem: the way to identify whether a problem is infeasible is using the two-phase method. If, in the phase I, it is not possible to eliminate the artificial variables from the problem and the optimal objective function value of phase I is not 0, then the original problem is infeasible.

- Unbounded problem: we remember that when deciding the leaving basic variable we looked at pivot column of x_t: $y_t = B^{-1}a_t$
  The LBV x_s was decided as:

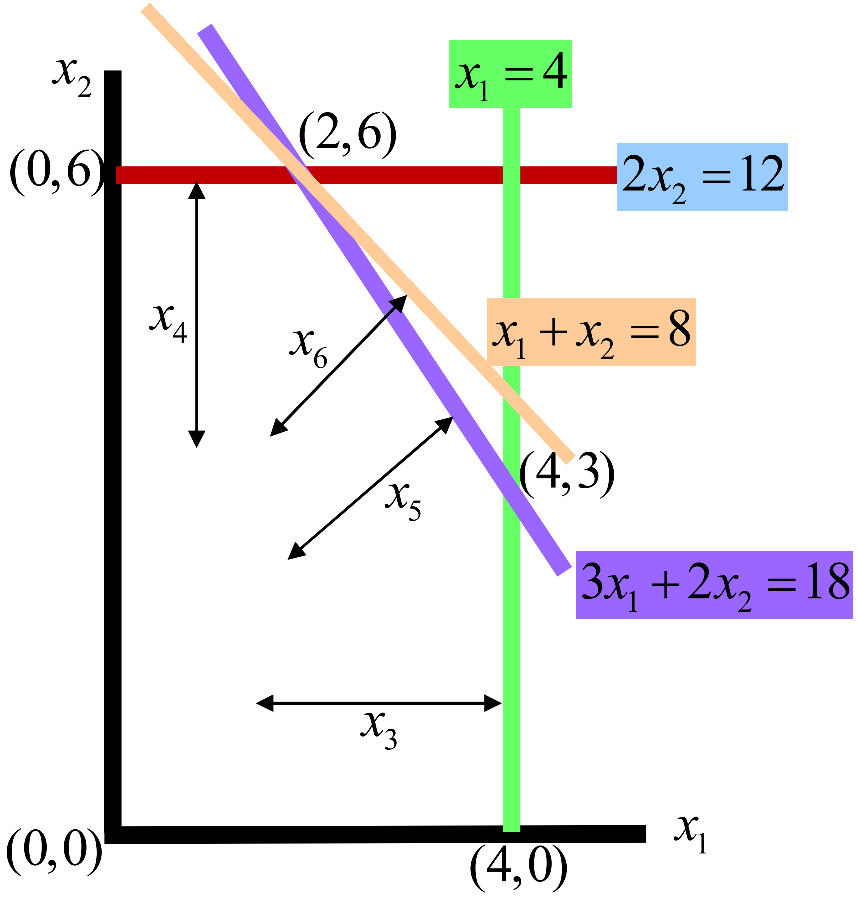
$$\frac{\hat{b}_s}{y_{st}} = \min_{1 \le i \le m} \left\{ \frac{\hat{b}_i}{y_{it}} : y_{it} > 0 \right\}$$

If all the $y_{it} \le 0$ the problem is unbounded

Degeneracy

- Example: Variable x_6 is degenerate (basic variable with value 0)

$$
\begin{array}{llllllll}
\min z = -3x_1 - 5x_2 \\
x_1 & & +x_3 & & & & = 4 \\
 & 2x_2 & & +x_4 & & & = 12 \\
3x_1 & +2x_2 & & & +x_5 & & = 18 \\
x_1 & +x_2 & & & & +x_6 & = 8 \\
x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & \geq 0
\end{array}
$$



DEGENERATE basic feasible solution

Some of the m basic variables = 0.

5

Duality

Duality (i)

PRIMAL $(m \times n)$	DUAL $(n \times m)$
$\max_x z = c^T x$ $Ax \leq b$ $x \geq 0$	$\min_y w = b^T y$ $A^T y \geq c$ $y \geq 0$
$\max_{x_j} z = \sum_{j=1}^{n} c_j x_j$ $\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m$ $x_j \geq 0 \quad j = 1, \ldots, n$	$\min_{y_i} w = \sum_{i=1}^{m} b_i y_i$ $\sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad j = 1, \ldots, n$ $y_i \geq 0 \quad i = 1, \ldots, m$

Duality (ii)

PRIMAL ($m \times n$)	DUAL ($n \times m$)
$\max z = 1.6x_1 + 1.4x_2$ $subject\ to:\ 10x_1 + 20x_2 \leq 8000$ $15x_1 + 10x_2 \leq 6000$ $18x_1 + 6x_2 \leq 6300$ $x_1, x_2 \geq 0$	$\min w = 8000\,y_1 + 6000\,y_2 + 6300\,y_3$ $subject\ to:\ 10\,y_1 + 15\,y_2 + 18\,y_3 \geq 1.6$ $20\,y_1 + 10\,y_2 + 6\,y_3 \geq 1.4$ $y_1, y_2, y_3 \geq 0$
$\max z = \begin{bmatrix} 1.6 & 1.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{bmatrix} 10 & 20 \\ 15 & 10 \\ 18 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 8000 \\ 6000 \\ 6300 \end{bmatrix}$ $x_1, x_2 \geq 0$	$\min w = \begin{bmatrix} 8000 & 6000 & 6300 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $\begin{bmatrix} 10 & 15 & 18 \\ 20 & 10 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 1.6 \\ 1.4 \end{bmatrix}$ $y_1, y_2, y_3 \geq 0$

Duality (iii)

TRANSFORMATION TABLE

min		max
Variable		**Constraint**
≥ 0	⟺	$\leq$
≤ 0	⟺	$\geq$
Free variable	⟺	$=$
Constraint		**Variable**
$\geq$	⟺	≥ 0
$\leq$	⟺	≤ 0
$=$	⟺	Free variable

Duality (iv)

Theorems and properties:

"The dual of the dual problem is the primal problem"

Weak duality theorem:

If x is a feasible solution of the primal and y is a feasible solution of the dual, then the following is true: $c^T x \leq b^T y$

Strong duality theorem :

If $\hat{x}$ is an optimal solution of the primal and $\hat{y}$ is optimal for the dual, Then the following is true: $c^T \hat{x} = b^T \hat{y}$

Duality (v)

Complementarity property of slack variables

If an optimal solution of the PRIMAL has slack or excess variable >0, then in the complementary DUAL solution, the dual variable associated with this constraint is zero and vice versa.

Duality Theorem

- If one problem has optimal feasible solutions, then its dual also has them
- If a problem has feasible solutions and the objective function is unbounded, then the other problem does not have feasible solutions
- If one problem does not have feasible solutions, then the other is either unbounded or does not have feasible solutions

Duality (vi)
Economic interpretation

- Shadow price (dual variable, simplex multiplier, equilibrium price, just price) of resource/constraint i (y_i) measures the marginal value of the resource, i.e., the marginal increase of objective function z when increasing marginally the resource/right-hand side b_i.

Source: OMIE

Duality (vii)
Relation between reduced costs and dual variables

Dual Problem	Primal Problem
The reduced costs of the slack and excess variables of the DUAL	are the optimal solutions of the variables of the PRIMAL
The reduced costs of the variables of the DUAL	are the optimal solutions of the slack and excess variables of the PRIMAL
The optimal solutions of the variables of the DUAL	are the reduced costs of the slack and excess variables of the PRIMAL

6

Sensitivity Analysis

Sensitivity Analysis (i)

Starting from the obtained optimal solution (in GAMS), we study how changing constraints, parameters and variables affect the optimal solution of the original problem.

Some examples of what we can alter in the original problem

① Change in constraints, right-hand side, or coefficients

② Change in the objective function

③ Addition of a new constraint

④ Addition of a new variable

Sensitivity Analysis (ii)

Firstly, we are carrying out numerical sensitivity analysis. Hence, the first step is to *code* the optimization problem, for example, in GAMS.

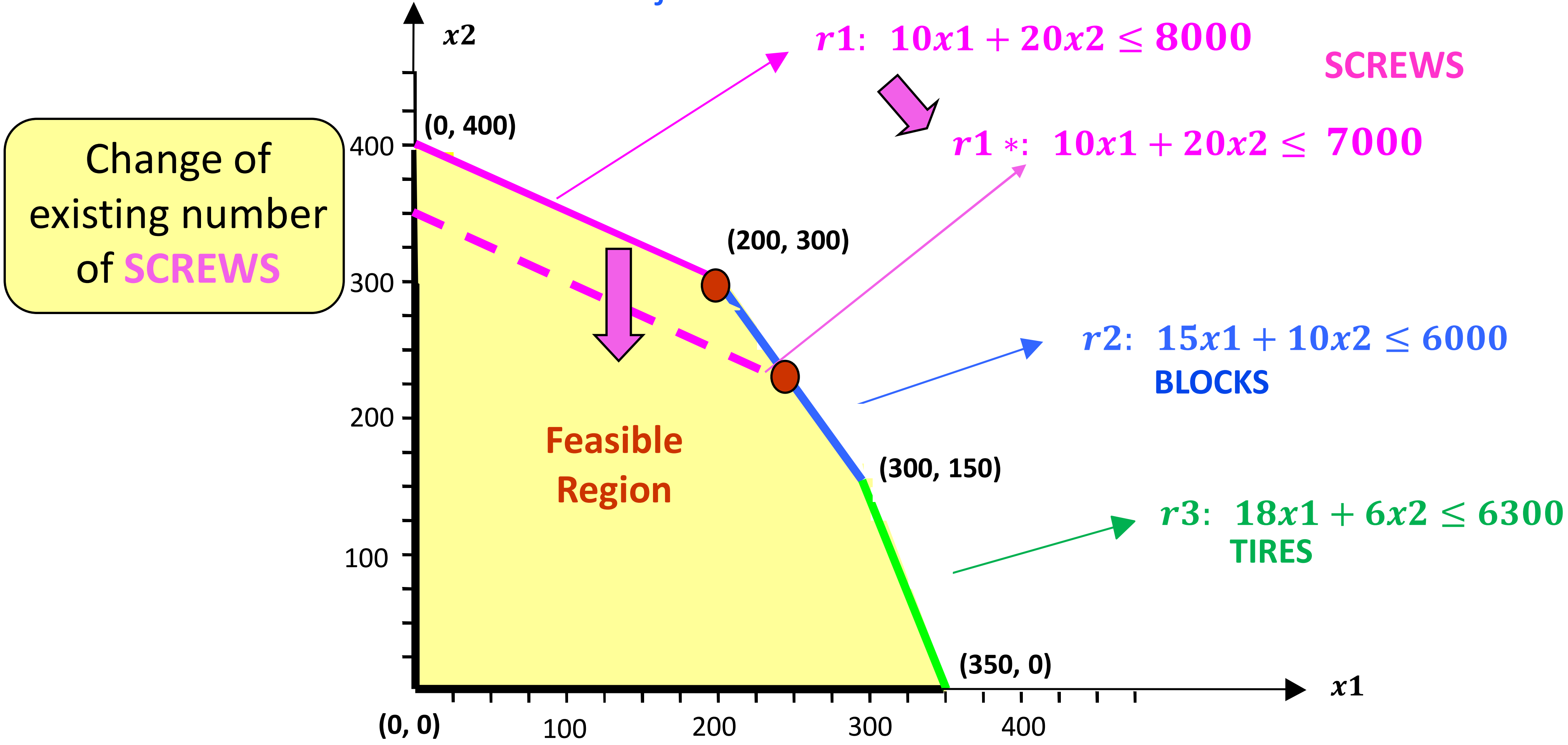
Below we consider the craftsman problem in GAMS code.

```
VARIABLES
VFO       value of the objective function: maximization of total profits
X1        number of produced trains
X2        number of produced trucks

POSITIVE VARIABLES X1,X2;

EQUATIONS
FO        objective function
SCREWS    limitation of screws
BLOCKS    limitation of blocks
TIRES     limitation of tires;
FO..           VFO =E= 1.6*X1 + 1.4*X2;
SCREWS..       10*X1 + 20*X2 =L= 8000;
BLOCKS..       15*X1+10*X2 =L= 6000;
TIRES..        18*X1+6*X2 =L= 6300;


MODEL TRAINSTRUCKS /ALL/

SOLVE TRAINSTRUCKS USING LP MAXIMIZING VFO
```

Sensitivity Analysis (iii)

Changes in right-hand side of constraints

- Can affect:
  - the feasibility of the previously obtained optimal solution
  - the value of the objective function



Sensitivity Analysis (iv)

Changes in right-hand side of constraints

Change the right-hand side value of the SCREWS constraint to 7000 and solve the model again.

```
VARIABLES
VFO       value of the objective function: maximization of total profits
X1        number of produced trains
X2        number of produced trucks

POSITIVE VARIABLES X1,X2;

EQUATIONS
FO        objective function
SCREWS    limitation of screws
BLOCKS    limitation of blocks
TIRES     limitation of tires;
FO..           VFO =E= 1.6*X1 + 1.4*X2;
SCREWS..       10*X1 + 20*X2 =L= 7000
BLOCKS..       15*X1+10*X2 =L= 6000;
TIRES..        18*X1+6*X2 =L= 6300;


MODEL TRAINSTRUCKS /ALL/

SOLVE TRAINSTRUCKS USING LP MAXIMIZING VFO
```

The new optimal solution yields an o.f. value of 715€.

Sensitivity Analysis (v)

Changes in right-hand side of constraints (cont.)

The impact of changes in the right-hand side of constraints can also be obtained by looking at the dual variables of these constraints.
The value of the dual variables of a constraint can be found in the .lst file once GAMS has solved the model. In particular, in the SolEQN listing.

	LOWER	LEVEL	UPPER	MARGINAL
EQU FO	.	.	.	1.000
EQU SCREWS	-INF	8000.000	8000.000	0.025
EQU BLOCKS	-INF	6000.000	6000.000	0.090
EQU TIRES	-INF	5400.000	6300.000	.

- 0.025(8000-7000)=25 → VFO=740-25=715

Even **without resolving the optimization problem**, the new value of the objective function can be obtained.