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Game Theory

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Contents

Introduction
Two-person zero-sum game with pure strategies
Two-person zero-sum game with mixed strategies
Cournot and Bertrand Equilibria

1

Introduction

Introduction

- Opposite to Decision Theory
- Conflict of interest and competitors
- Goal achievement depends not only on one's own decisions and uncertainties but also on competitors' decisions
- Rational criteria applied to strategy selection for your interest

Classification of Game Theory (i)

- Number of players
  - Two persons
  - N-persons
- Number of strategies
  - Finite
  - Infinite
- Time evolution
  - Static
  - Dynamic
- Information exchange among players
  - Cooperative
  - Noncooperative

Classification of Game Theory (ii)

- Wealth variation
  - Zero sum
  - Nonzero sum
- Available information from other players
  - Complete
  - Incomplete
- Amount of information got along the game
  - Perfect
  - Imperfect

Payoff Matrix

- X strategies of player 1 (finite)
- Y strategies of player 2 (finite)

Payoff matrix

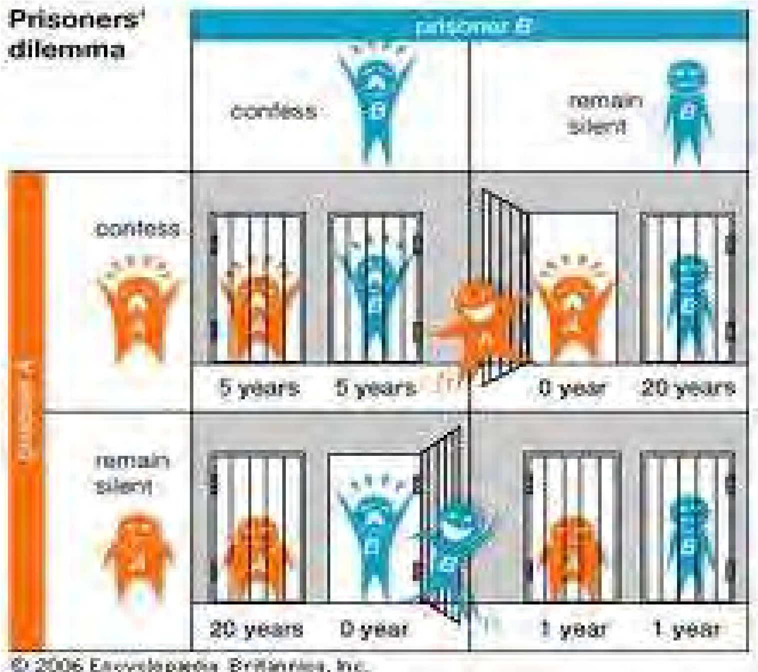
	J2		
J1	(a_{11}, b_{11})		
		(a_{ij}, b_{ij})	
			(a_{mn}, b_{mn})

- a_{ij} payment that player 1 receives if he/she selects the strategy i and player 2 selects the strategy j

- b_{ij} payment that player 2 receives if he/she selects strategy j and player 1 selects strategy i

Prisoner's Dilemma

- Two-person, finite, static, noncooperative, nonzero-sum game
- Two criminals are arrested and imprisoned as they are charged with a crime. Each prisoner is in solitary confinement with no means of speaking to or exchanging messages with the other. If both confess, each of them serves 5 years in prison. If one confesses and the other remains silent, the one who confesses will be free, and the other will serve 20 years in prison. If both remain silent, both will serve 1 year in jail.



		J2	
		Confess	**Remain silent**
J1	**Confess**	(-5,-5)	(0,-20)
	Remain silent	(-20,0)	(-1,-1)

Prisoner's Dilemma

- Equilibrium point: the best decision that both players can make, taking into account the decisions of the other in a context of economic rationality

		J2	
		Confess	Remain silent
J1	Confess	(-5 , -5)	(0 , -20)
	Remain silent	(-20 , 0)	(-1 , -1)

- Equilibrium point when both "best strategies" match
- We won't always find an equilibrium
- Equilibrium point does not imply optimal solution (as in this case)
  - If the game were cooperative (Exchange Information between players), the equilibrium would be (Remain Silent, Remain Silent) *if you trust*

Equilibrium Strategies (Nash)

- Not always there is an Equilibrium Strategy.

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	(0 , 0)	(0 , 1)	(1 , 0)
	Paper	(1 , 0)	(0 , 0)	(0 , 1)
	Scissors	(0 , 1)	(1 , 0)	(0 , 0)

- And it can be non-unique: Correlated Equilibrium.

		Woman	
		Football	TV Show
Man	Football	(2 , 1)	(0 , 0)
	TV Show	(-1 , -1)	(1 , 2)

John F. Nash *Non-Cooperative Games* PhD Thesis. Princeton University. May 1950

2

Two-person zero-sum game with pure strategies

Solving two-person zero-sum game

1. Search for *pure strategies* in equilibrium. If they exist, do not continue to step 2.
   – Pure strategy: unique strategy
2. Solve the problem with *mixed strategies* using LP.
   – Mixed strategy: mix of strategies with specific probabilities

- At any moment, eliminating dominated strategies can reduce the payoff matrix.

Two-person zero-sum game

- These games are noncooperative.
- It is only necessary to provide the payoff matrix of one player.
- Each player should expect the worst strategy from the other player.
- The equilibrium is the optimal solution of the game.
- Payoff matrix from the point of view of payments to player 1.

			J 2	
		E'1	E'2	E'3
	E1	(1,-1)	(2,-2)	(4,-4)
J 1	E2	(1,-1)	(0,0)	(5,-5)
	E3	(0,0)	(1,-1)	(-1,1)

			J 2	
		E'1	E'2	E'3
	E1	1	2	4
J 1	E2	1	0	5
	E3	0	1	-1

Removal of dominated strategies (i)

- *Dominated Strategy*
  - A strategy of one player that is always worse or equal to another independently of the chosen strategy of the other player.
- Example: strategy E3 of player 1 is removed

			J 2	
		E'1	E'2	E'3
	E1	1	2	4
J 1	E2	1	0	5
	E3	0	1	-1

			J 2	
		E'1	E'2	E'3
	E1	1	2	4
J 1	E2	1	0	5

- Strategy E'3 of player 2 is removed

			J 2	
		E'1	E'2	E'3
	E1	1	2	4
J 1	E2	1	0	5

			J 2	
		E'1	E'2	
	E1	1	2	
J 1	E2	1	0	

Removal of dominated strategies (ii)

- Strategy E2 of player 1 is removed

			J 2	
		E'1	E'2	
	E1	1	2	
J 1	E2	1	0	

			J 2	
		E'1	E'2	
	E1	1	2	
J 1				

- Strategy E'2 of player 2 is removed

			J 2	
		E'1	E'2	
	E1	1	2	
J 1				

			J 2	
		E'1		
	E1	1		
J 1				

- Then, strategy E1 of player 1 and strategy E'1 of player 2 are in equilibrium

Game Value and Fair Game

- Game Value
  - Payment to player 1 when both players play optimally.
  - Example: game value is 1

- Fair Game
  - When the game value is 0

Minimax Criterion

- Minimize the maximum loss (risk aversion criterion)

		J 2			Minima	
		E’1	E’2	E’3		
J 1	E1	-3	-2	6	-3	
	E2	2	0	2	0	Maximin
	E3	5	-2	-4	-4	
Maxima		5	0	6		
			Minimax			

- *Maximin*: maximum of minimum payments to player 1
- *Minimax*: minimum of maximum losses of player 2
- If minimax and maximin are equal in one combined strategy, then the equilibrium is stable, and this is an equilibrium point.

Non-stable Solution: No equilibrium

		J 2			Minima	
		E'1	E'2	E'3		
J 1	E1	0	-2	2	-2	Maximin
	E2	5	4	-3	-3	
	E3	2	3	-4	-4	
Maxima		5	4	2		
				Minimax		

3

Two-person zero-sum game with mixed strategies

Messi vs. Courtois

- Messi has to take a penalty against Courtois. He can choose to shoot to the left or the right

- Courtois has to decide, before Messi takes the shot, whether to move to the left or the right.

- The probability of Messi scoring a goal depends on his and Courtois' decisions.

		Courtois	
		Left	Right
Messi	Left	55%	90%
	Right	80%	60%

- What would be the strategy of each player?

Games with mixed strategies (i)

- Each possible strategy of each player has an associated probability.
  - x_i probability of player 1 to use the strategy i, $i = 1, \dots, m$

$$\sum_{i=1}^{m} x_i = 1$$

  - y_j probability of player 2 to use the strategy j, $j = 1, \dots, n$

$$\sum_{j=1}^{n} y_j = 1$$

  - p_{ij} payment to player 1 if he/she selects the strategy i and the player 2 selects the strategy j
- Expected payment to player 1: $\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j p_{ij}$
- v game value
- $\underline{v}$ maximin for player 1
- $\bar{v}$ minimax for player 2

Games with mixed strategies (ii)

- Minimax criterion for mixed strategies
  - A player should select the mixed strategy that minimizes the maximum expected loss.

- Minimax theorem
  - If mixed strategies are allowed, the optimal strategies using the minimax criterion provide a stable solution if $v = \underline{v} = \bar{v}$, no player wants to change his/her strategy unilaterally.

Solution using linear programming

- Maximize the minimum expected payment to player 1

$$\max v$$

- For any pure strategy of player 2

$$\sum_{i=1}^{m} x_i p_{ij} \geq v \qquad \forall j = 1, \dots, n$$

- The sum of probabilities is 1

$$\sum_{i=1}^{m} x_i = 1$$

$$x_i \geq 0 \quad \forall i = 1, \dots, m$$

- Expected payment to player 1

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j p_{ij}$$

We will get the dual problem if analyzed from player 2 point of view

Player 2 minimizes its expected losses based on the decisions of player 1

$$\max v$$
$$\sum_{i=1}^{m} x_i p_{ij} \geq v \quad \forall j = 1, \dots, n$$
$$\sum_{i=1}^{m} x_i = 1$$
$$x_i \geq 0 \quad \forall i = 1, \dots, m$$

Player 1 maximizes its expected payments based on the decisions of player 2

$$\min w$$
$$\sum_{j=1}^{n} y_j p_{ij} \leq w \quad \forall i = 1, \dots, m$$
$$\sum_{j=1}^{n} y_j = 1$$
$$y_j \geq 0 \quad \forall j = 1, \dots, n$$

- These two formulations are dual. It is only necessary to solve one point of view. At the optimum, $v^* = w^*$

Solution to the problem

Messi maximizes its expected payments based on the decisions of Courtois choosing

ML: 36%

MR: 64%

Courtois maximizes its expected payments based on the decisions of Messi's choosing

CL: 55%

CR: 45%

The expected payment for Messi would be

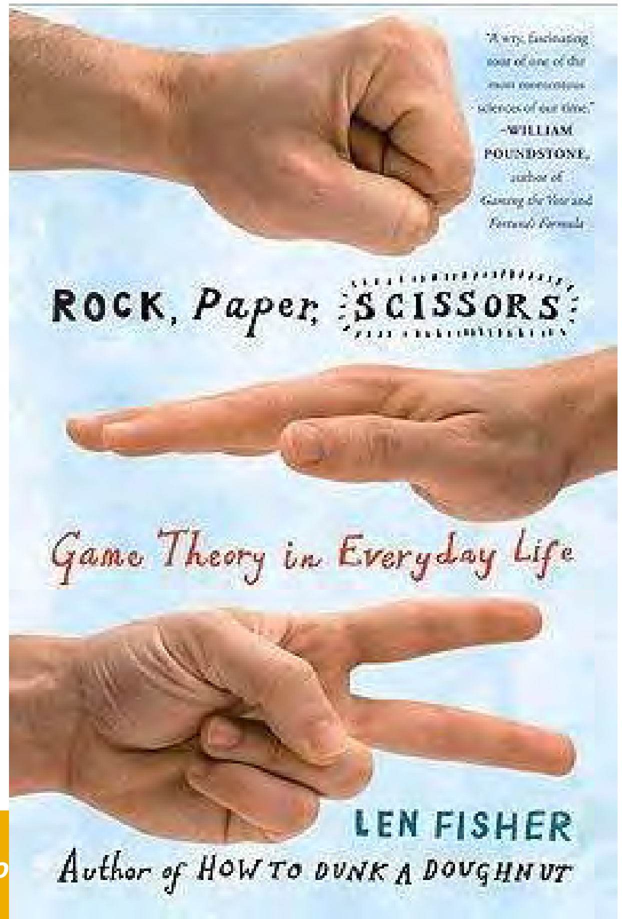
$$\sum_{i=1}^{2}\sum_{j=1}^{2} x_i y_j p_{ij} = 0.709$$

Messi following the strategy 36% left/64% right will score in 71% of occasions

Example of two-person zero-sum game

- There are no dominated strategies. There is no equilibrium with pure strategies.

			J 2			Min
		Paper	Rock	Scissors		
	Paper	0	1	-1		-1
J 1	Rock	-1	0	1		-1
	Scissors	1	-1	0		-1
Max		1	1	1		



Example. Formulation from point of view of player 2

$$
\begin{array}{lllll}
\min w & & & & \\
 & y_2 & -y_3 & \leq & w \\
-y_1 & & +y_3 & \leq & w \\
y_1 & -y_2 & & \leq & w \\
y_1 & +y_2 & +y_3 & = & 1 \\
y_i \geq 0 & & & &
\end{array}
$$

$$(y^*_1, y^*_2, y^*_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$(\pi^*_1, \pi^*_3, \pi^*_3) = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{-1}{3}\right)$$

- Change the sign of dual variables $(x^*_1, x^*_2, x^*_3) = (1/3,1/3,1/3)$
- Game value is 0
- For each player, the mixed strategy in equilibrium consists of selecting each strategy with equal probability.

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Cournot and Bertrand Equilibria

4

Cournot and Bertrand Equilibria

Elasticity on the price – Monopoly

- We have a company operating in monopoly producing and selling a product

- The price of the product is elastic

$$P(Q) = \begin{cases} a - Q & Q < a \\ 0 & Q > a \end{cases}$$

Price sensitivity: coefficient of Q (-1)

- The unitary production cost is c

- A company in a monopoly has to decide how much to produce

Profit = $(P(Q) - c)Q$

Profit = $(a - Q - c)Q = (a - c)Q - Q^2$

$$\frac{dProfit}{dQ} = a - c - 2Q = 0$$

$$Q = (a - c)/2$$

Price=$\frac{(a+c)}{2}$

Profit=$\frac{(a-c)^2}{4}$

Duopoly – Cournot equilibrium

- Two companies compete with a homogeneous product (i.e., there is no difference based on the producer)
- Both companies do not cooperate, and the competition is done once (static game)
- The price of the product is elastic

$$P(Q) = \begin{cases} a - Q & Q < a \\ 0 & Q > a \end{cases}$$

- The unitary production cost is equal for both companies c
- The total volume is $Q = q_1 + q_2$

Profit

$$\pi_i(q_1, q_2) = q_1[P(q_1 + q_2) - c] = q_1[a - (q_1 + q_2) - c]$$

For player 1

$$\max \pi_1(q_1, q_2^*) = \max q_1[a - (q_1 + q_2^*) - c]$$

$$\frac{\partial \pi_1(q_1, q_2^*)}{\partial q_1} = a - 2q_1 - q_2^* - c = 0$$

$$q_1^* = \frac{(a - q_2^* - c)}{2}$$

In the equilibrium

$$q_1^* = q_2^* = \frac{a-c}{3} \qquad Q^* = \frac{2}{3}(a-c)$$

$$P(Q) = a - \frac{2}{3}(a-c) = \frac{a+2c}{3}$$

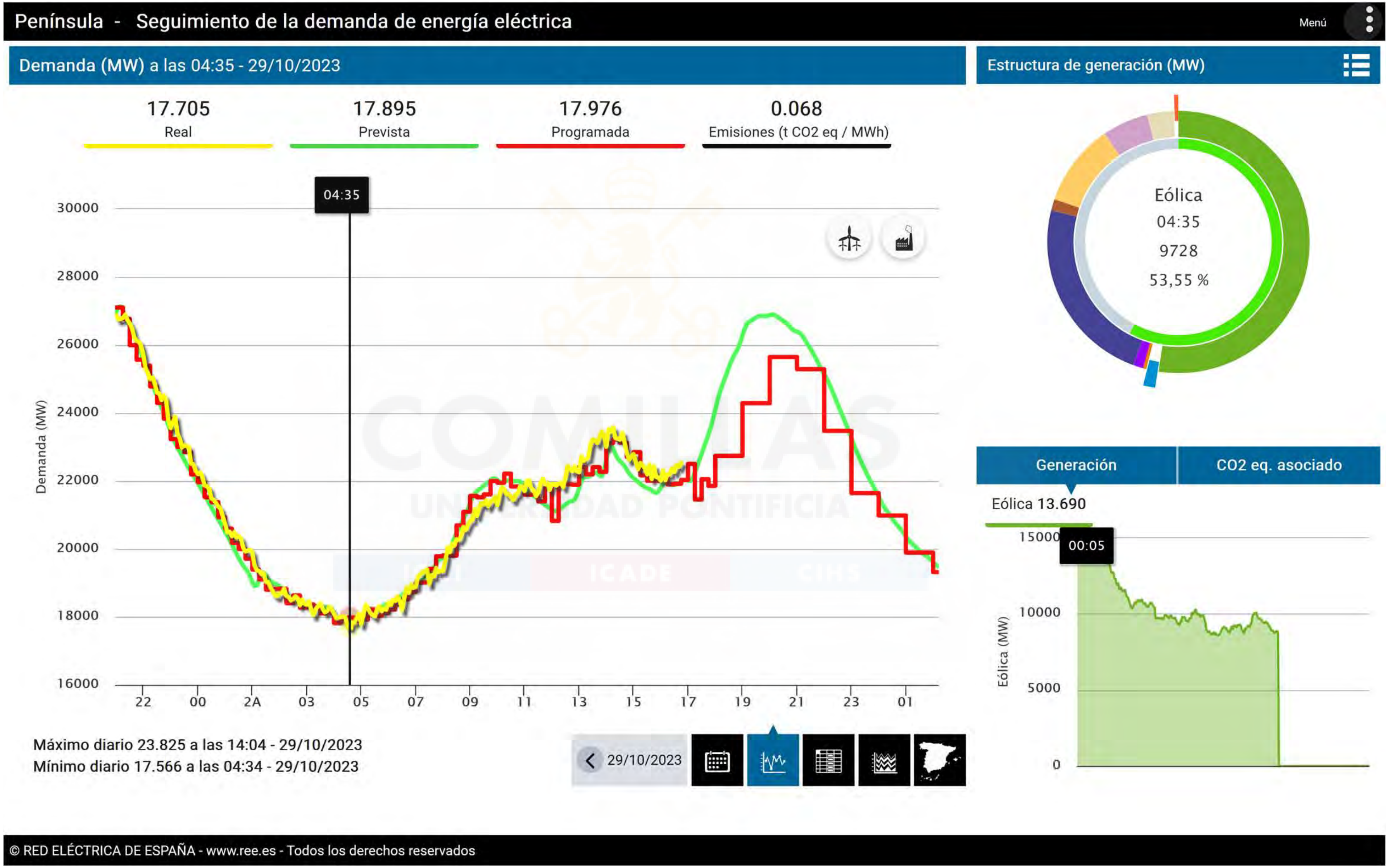
$$\pi_i + \pi_2 = (q_1 + q_2)(P - c) = \frac{2(a-c)^2}{9}$$

Comparison of monopoly and duopoly

	Monopoly	Duopoly
Total production	$\frac{1}{2}(a-c)$	$\frac{2}{3}(a-c)$
Price	$\frac{a+c}{2}$	$\frac{a+2c}{3}$
Total profit	$\frac{(a-c)^2}{4}$	$\frac{2(a-c)^2}{9}$

- Incentive to collude (to act like a cartel) $(a-c)^2/36$
- Given that cartels are illegal, agents try to collude tacitly using strategies of decreasing production volume by themselves, whose effects are price rising and net profit increases for them.

Spanish electricity demand on Oct 29, 2023



Bertrand equilibrium

- Companies produce different products
- They can choose the selling price: p_i price for the company i
- The product demand is elastic
  $q_i(p_i, p_j) = a - p_i + bp_j$
- Unitary production cost c is equal for both companies
- Space of strategies of each company

- Payoff matrix (net profits)

$$\pi_1(p_1, p_2) = q_1(p_1, p_2)[p_1 - c] = (a - p_1 + bp_2)(p_1 - c)$$

- Maximize the payment function for the company i

$$\max_{p_1 \geq 0} \pi_1(p_1, p_2^*) = \max_{p_1 \geq 0} (a - p_1 + bp_2^*)(p_1 - c)$$

$$\frac{\partial \pi_1(p_1, p_2^*)}{\partial p_1} = -p_1 + c + a - p_1 + bp_2^* = 0$$

$$p_1^* = \frac{1}{2}(a + bp_2^* + c)$$

- Equilibrium point

$$p_1^* = p_2^* = \frac{a + c}{2 - b} \quad (b < 2)$$

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