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BENDERS DECOMPOSITION FOR MIXED-INTEGER HYDROTHERMAL PROBLEMS BY LAGRANGEAN RELAXATION

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PSCC 02

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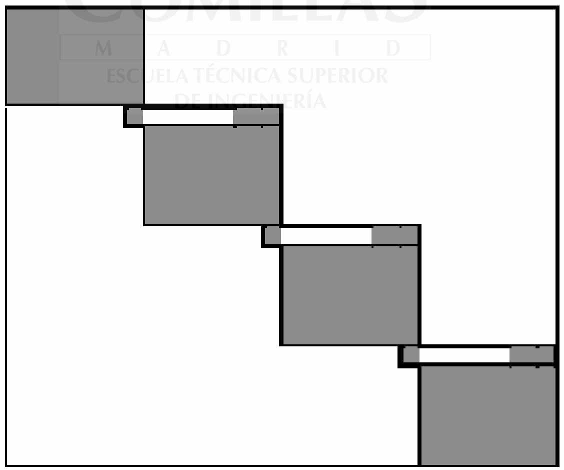
Hydrothermal coordination problem

- Minimization of total operation cost subject to demand supply.
- Multiperiod problems usually present a staircase matrix structure suitable to be solved via a Benders decomposition algorithm.



Mixed integer variables

- Appear when modeling nonlinear curves (e.g., PQH curves) as a piecewise functions
- Mixed integer variables that represent commitment, start up and shut down of thermal units

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Benders decomposition review (I)

- Resolution of an LP problem with the form

$$
\begin{aligned}
&\min z = c^1 x + c^2 y \\
&A^{11} x \qquad\qquad = b^1 \\
&A^{21} x + A^{22} y = b^2 \\
&x \geq 0, y \geq 0; x, y \in \mathbb{R}^n
\end{aligned}
$$

via formulating a master problem that considers all ‘first stage’ variables *x* and a subproblem in the remaining variables *y*

Benders decomposition review (II)

- LP master Problem (MP)

$$\min z = c^1 x + \theta(x)$$
$$A^{11} x \;= b^1$$
$$x \geq 0, x \in V$$

- LP subproblem (S)

$$\theta(x) = \left\{ \min c^2 y \,/\, A^{22} y = b^2 - A^{21} x \,/\, y \geq 0 \right\}$$

with

$$V = \left\{ x \,/\, \theta(x) < \infty \right\}$$

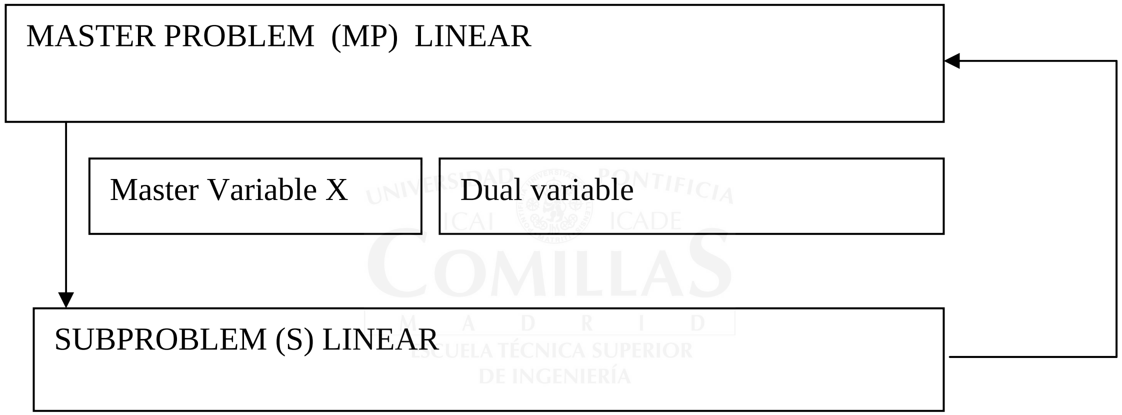
Benders decomposition review (III)

- Master problem MP is solved by sequentially outer approximating the recourse function $\theta(x)$ and the set *V*. This outer approximations define a relaxed master problem (RMP) with the form

$$
\begin{aligned}
&\min z = c^1 x + \theta \\
&A^{11} x = b^1 \\
&0 \geq \sigma^k (b^2 - A^{21} x) \quad k = 1, \ldots, K \quad \text{feasibility cut} \\
&\theta \geq \pi^j (b^2 - A^{21} x) \quad j = 1, \ldots, J \quad \text{optimality cut} \\
&x \geq 0
\end{aligned}
$$

Benders decomposition review (IV)

- The Benders decomposition algorithm is summarized on the following picture



Benders decomposition review (V)

- Benders algorithm is based on the linear duality theorem that guarantees the solution of the primal problem to be equal to the solution of the dual problem.
- In particular this leads to the convexity of the recourse function.

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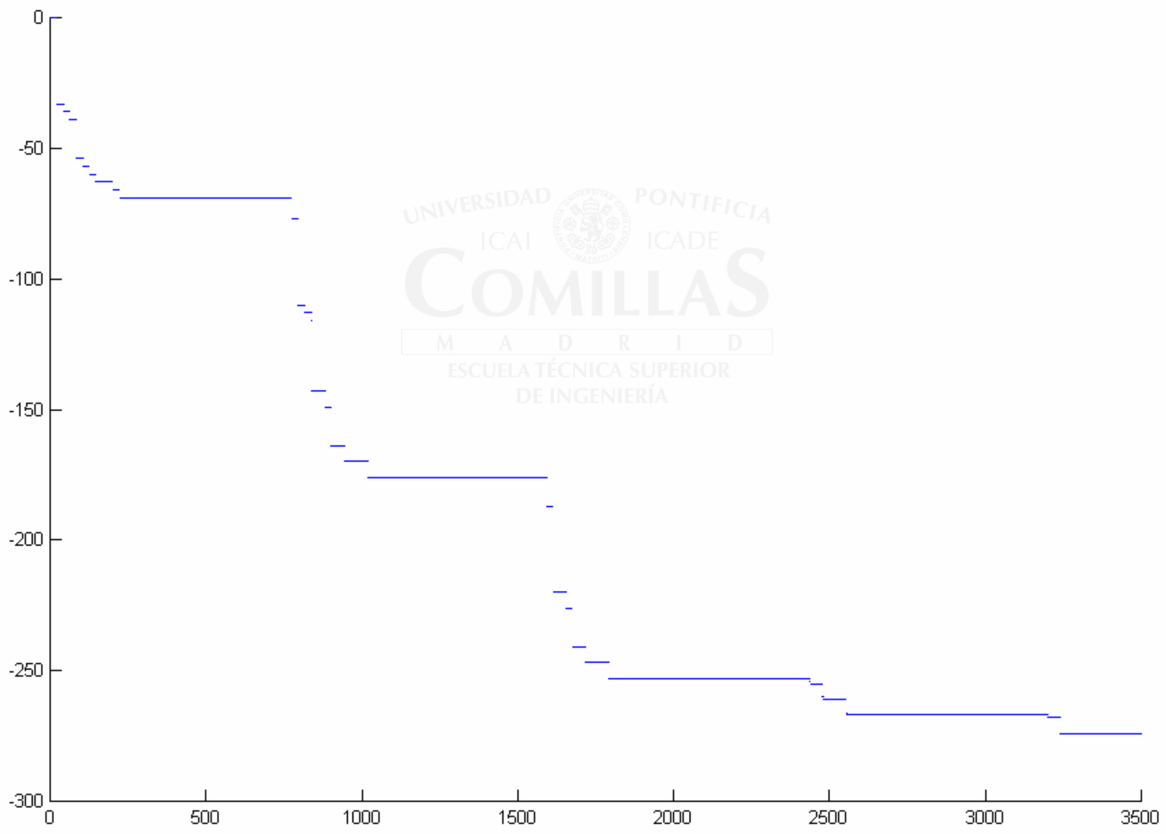
Extension to mixed integer variables (I)

- Example of non convex recourse function for a MIP problem

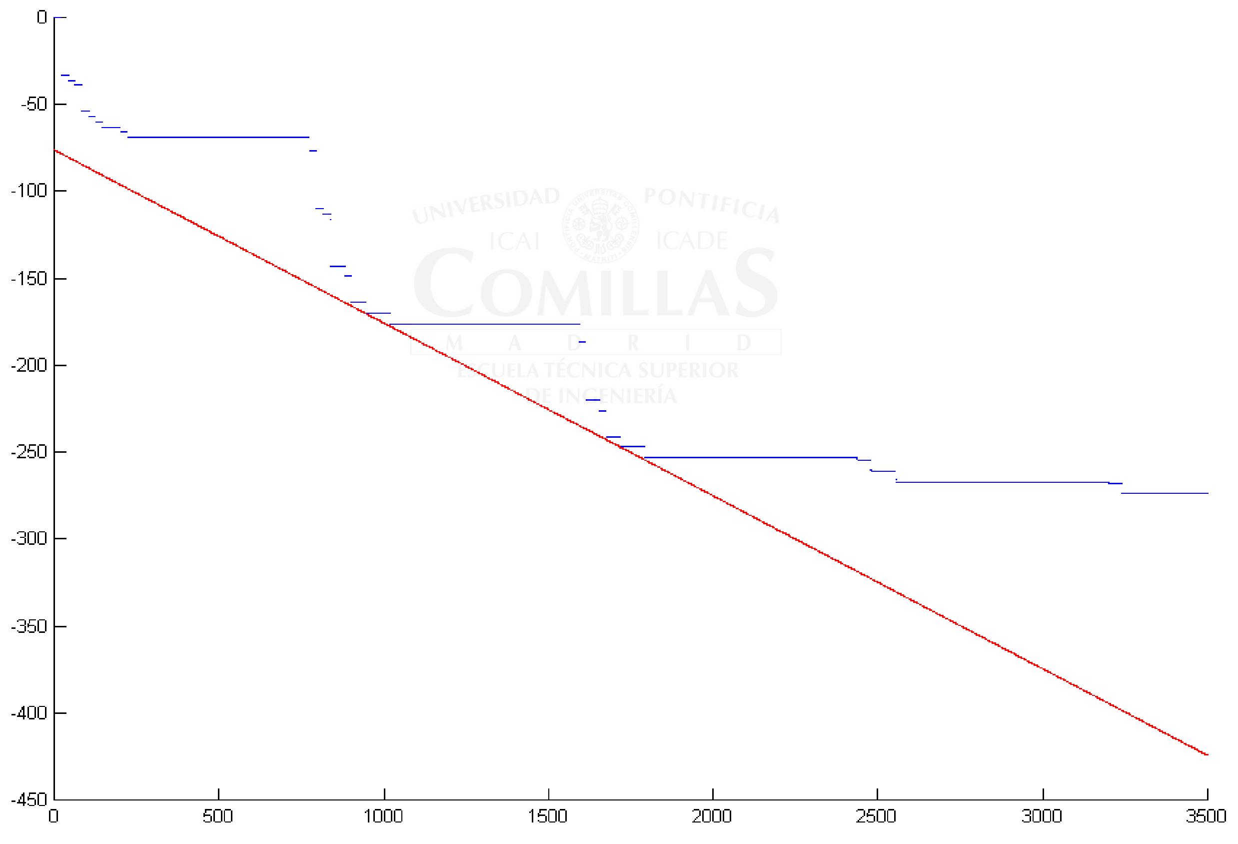


Extension to mixed integer variables (II)

- Using Geoffrion results a partial convexification of the recourse function is obtained using a Lagrangean relaxation algorithm to solve the Benders subproblem.

Extension to mixed integer variables (III)

- Example of non convex recourse function for a MIP problem



Extension to mixed integer variables (IV)

- Using Geoffrion results a partial convexification of the recourse function is obtained using a Lagrangean relaxation algorithm to solve the Benders subproblem.
- The relaxed equations are those that connect first and second stage variables.
- In a hydrothermal coordination problem these are the equations of reserve management and the equations that represents start up and shut down variables for thermal units

$$R_{p-1} + I_p - P_p = R_p$$

$$c_p - c_{p-1} = a_p - p_p$$

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Lagrangean Relaxation (I)

1. To solve the MIP Benders Subproblem (S) via Lagrangean relaxation we formulate the dual problem (D) that consist of

$$(D) \quad \max \left\{ w(\lambda), \lambda \geq 0 \right\}$$

2. With Lagrangean subproblem (LS)

$$(LS) \quad \begin{gathered} w(\lambda) = \min c^2 y + \lambda \left(A^{22} y - b^2 + A^{21} x \right) \\ y \in Y, y \geq 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2} \end{gathered}$$

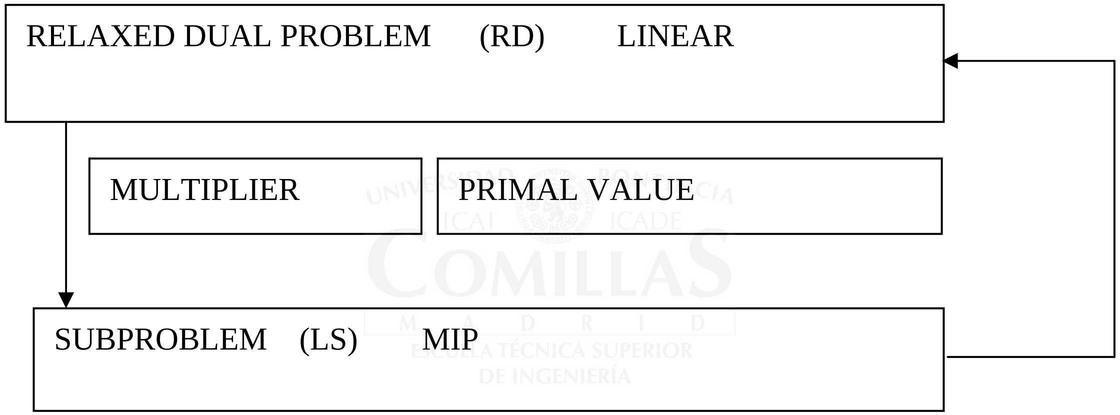
Lagrangean relaxation (II)

- The resolution of the dual problem (D) is carried out formulating a relaxed dual problem (RD) and a subproblem that outer approximates the dual function

$$
(RD)\quad
\begin{aligned}
&\max w\\
&w \le c^2 y^k + \lambda\left(A^{22} y^k - b^2 + A^{21} x\right)\\
&y^k \in \operatorname{conv}\left\{y \in Y, y \ge 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\}\\
&k : 1, \ldots, K
\end{aligned}
$$

Lagrangean Relaxation (III)

- Summary of the Lagrangean Relaxation algorithm

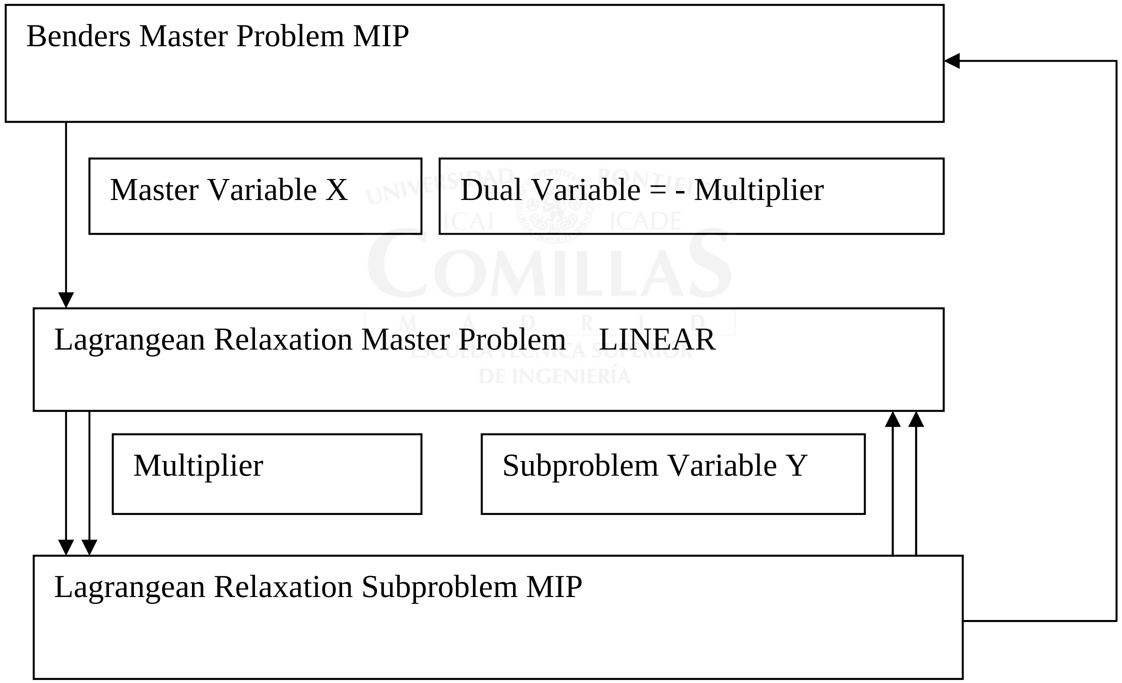


Lagrangean Relaxation (IV)

- Phase I
  - Determines the minimum number of Lagrangean cuts needed to have a bounded relaxed dual problem (RD).
  - In case of unboundness of the relaxed dual problem the lagrangean relaxation phase I finishes with the minimization of infeasibilities of the subproblem due to the coupling constraints

Lagrangean Relaxation (V)

- Benders decomposition with MIP subproblem



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Numerical Results (I)

- Results for a MIP deterministic 12 period problem.
- Coded in a modeling language
- Resolution via two stage decomposition

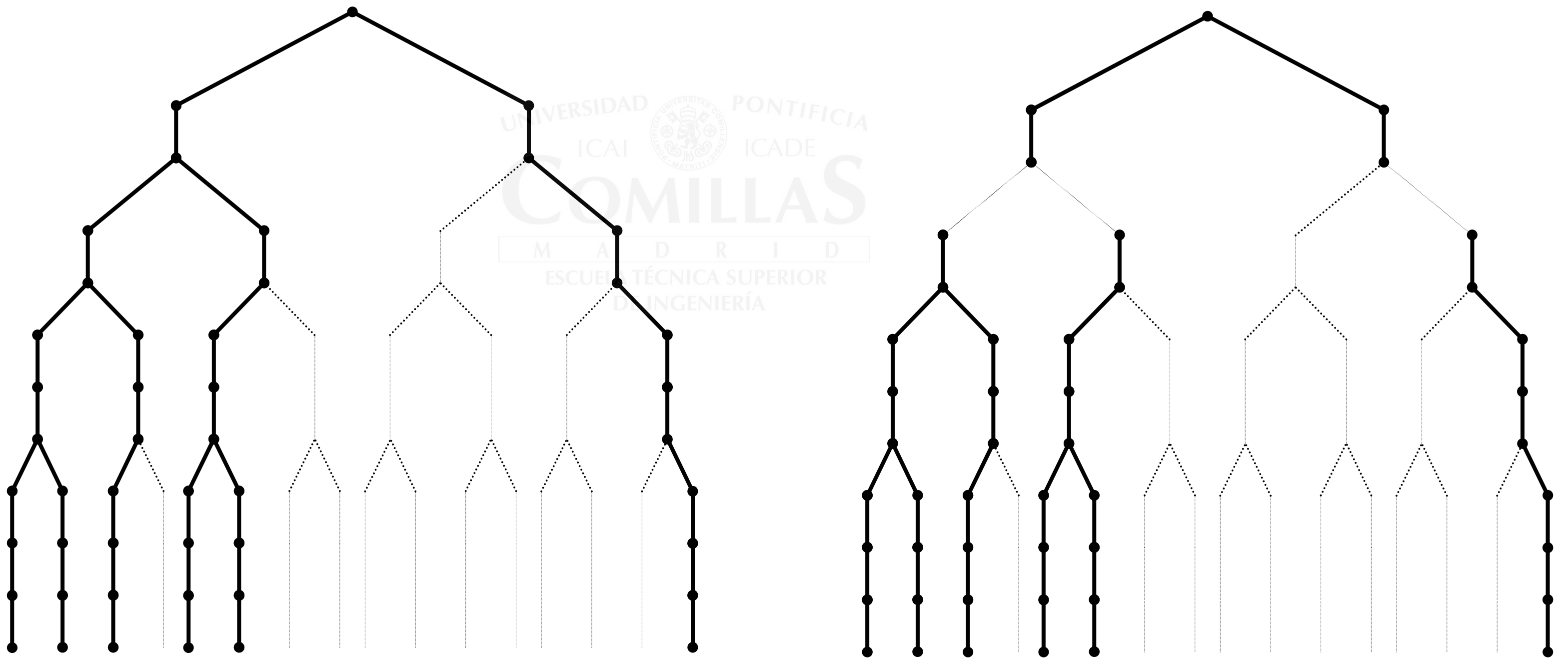
Master per.	Master size (r,v,n,b)				Subproblem size (r,v,n,b)				Bd Iter.	LR Iter.	Exec. time (s)	Solution time (s)
1	1605	2183	6767	146	17127	23374	72167	1606	11	166	3365	7438
1,2	3128	4265	13165	292	15604	21292	65769	1460	22	196	4646	10065
1,2,3	4644	6342	19569	438	14089	19215	59391	1314	18	176	2908	7184
1,2,3,4	6215	8489	26179	584	12517	17068	52755	1168	40	266	5209	15264
1,2,3,4,5	7714	10541	32476	730	11018	15016	46458	1022	42	243	2861	8327
1,2,3,4,5,6	9316	12721	39238	876	9416	12836	39696	876	4	266	1075	3673
1,..........,7	10906	14890	45950	1022	7826	10667	32984	730	5	304	1063	3860
1,...........8	12463	17019	52521	1168	6269	8538	26413	584	71	413	5426	32176
1,...........9	15598	21306	65747	1314	3134	4251	13187	438	94	336	2771	22737

Table 1. Iterations for a MIP fastpass traversing strategy.

- Numerical results of the last column due to the modeling language.

Numerical results (II)

- Extension to stochastic problems
- Resolution of a stochastic problem via subtree decomposition



Numerical Results (III)

- Resolution of a stochastic problem via subtree decomposition

Decomposition	Feasibility strategy			
	Bd It	Sol time	Exec time	Quality
1	OUT OF MEMORY			
1 2	2	408	294	3.2e-6
1 2 4	5	3258	2558	3.2e-6
1 2 4 6	17	2762	1056	158.1e-6
1 3 5	11	2554	1801	158.1e-6
1 2 3	5	648	340	80.6e-6

Table 4. Comparison among subtrees.

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Conclusions

- Extension of the Benders decomposition algorithm for MIP subproblems via the Lagrangean relaxation algorithm
- Use of decomposition methods only valuable for huge MIP problems
- Further research will focus on the development of this algorithm in C++ and on integration of Lagrangean relaxation algorithm and branch and bound.

Mixed integer variables (I)

- Appear when modeling nonlinear curves as PQH curves
- Traditional ways of modeling nonlinear curves comprise λ-form and δ- form

$$\lambda - form$$

$$x = x_1\lambda_1 + \cdots + x_n\lambda_n$$

$$y = y_1\lambda_1 + \cdots + y_n\lambda_n$$

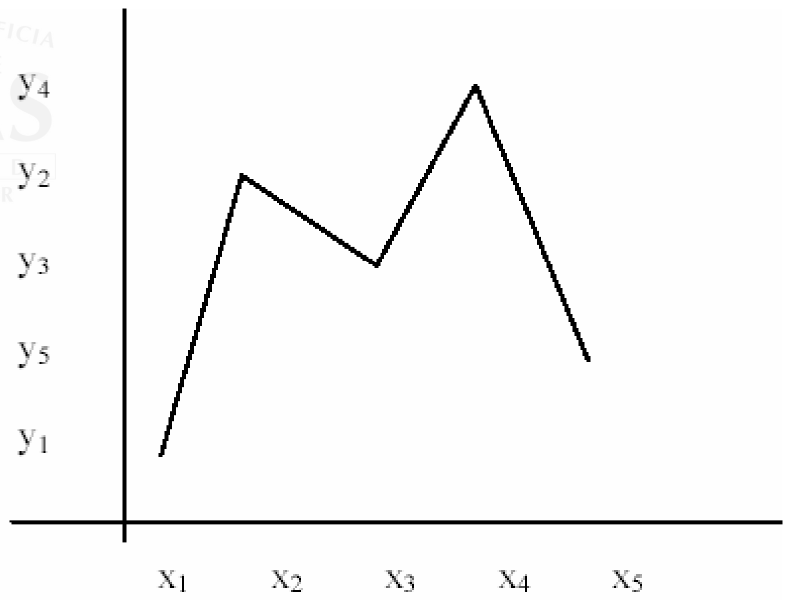
$$\lambda_1 + \cdots + \lambda_n = 1$$

$$\delta - form$$

$$x = (x_2 - x_1)\delta_1 + \cdots + (x_n - x_{n-1})\delta_{n-1}$$

$$y = (y_2 - y_1)\delta_1 + \cdots + (y_n - y_{n-1})\delta_{n-1}$$

$$0 \le \delta_1, \ldots, \delta_{n-1} \le 1$$



Mixed integer variables (II)

- λ- form
  - At most two adjacent λ s can be non-zero
  - This logical condition requires the use of extra binary variables to be modeled
- δ- form
  - if any δi is non-zero, all the preceding δi must take the value 1 and all the succeeding δi must take the value 0.
  - This logical condition requires the use of extra binary variables to be modeled

Mixed integer variables (III)

- Mixed integer variables that represent starts and stops of thermal groups and that are introduced in the objective function

$$c_p - c_{p-1} = a_p - p_p$$

$$0 \le a_p, p_p \le 1$$

$$c_p \in \{0,1\}$$

Lagrangean Relaxation (VI)