Max/Min Output Polytopes Including Startup and Shutdown Capabilities: The Case of The Unit Commitment Problem

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Nomenclature

Upper-case letters are used for denoting parameters and sets. Lower-case letters denote variables and indexes.

0.1. Indexes and Sets

$t \in \mathcal{T}$ Hourly periods, running from 1 to T hours.

0.2. Constants

$\overline{P}$ Maximum power output of unit g [MW].

$\underline{P}$ Minimum power output of unit g [MW].

SD Shutdown capability of unit g [MW].

SU Startup capability of unit g [MW].

0.3. Variables

0.3.1. Positive and Continuous Variables

p_t Power output at period t of unit g, production above the minimum output $\underline{P}_g$ [MW].

0.3.2. Binary Variables

u_t Commitment status of the unit g for period t, which is equal to 1 if the unit is online and 0 offline.

v_t Startup status of unit g, which takes the value of 1 if the unit starts up in period t and 0 otherwise.

w_t Shutdown status of unit g, which takes the value of 1 if the unit shuts down in period t and 0 otherwise.

$$
\begin{array}{ccccccc}
 & & u_{t-1} & u_t & v_t & p_{t-1} & p_t \\
x^{(1)} & = & (0 & 0 & 0 & 0 & 0) \\
x^{(2)} & = & (0 & 1 & 1 & 0 & 0) \\
x^{(3)} & = & (1 & 0 & 0 & 0 & 0) \\
x^{(4)} & = & (1 & 1 & 0 & 0 & 0) \\
x^{(5)} & = & (1 & 1 & 0 & \Delta & \Delta) \\
x^{(6)} & = & (1 & 1 & 0 & \Delta & 0) \\
x^{(7)} & = & (1 & 1 & 0 & 0 & \Delta)
\end{array}
$$

Fig. 1: I think that those are all the possible vertexes

1. Formulation with $SU = SD = \underline{P}$

1.1. Generation Limits

The total unit production is $\underline{P} \cdot u_t + p_t$: the minimum power output $\underline{P}$ that is generated just by being committed, and the generation over that minimum p_t. The generation limits over the power output and considering unit's startup and shutdown capabilities are set following the model proposed by Morales-España et al. [1]:

$$p_t \leq \left(\overline{P} - \underline{P}\right)(u_t - v_t) \qquad \forall t \in \mathcal{T} \quad (1)$$
$$p_{t-1} \leq \left(\overline{P} - \underline{P}\right)(u_{t-1} - w_t) \qquad \forall t \in [2, |\mathcal{T}|] \quad (2)$$
$$u_t - u_{t-1} = v_t - w_t \qquad \forall t \in [2, |\mathcal{T}|] \quad (3)$$
$$w_t \leq 1 - u_t \qquad \forall t \in [2, |\mathcal{T}|] \quad (4)$$
$$p_t \geq 0 \qquad \forall t \in \mathcal{T} \quad (5)$$
$$u_t \in \{0, 1\} \qquad \forall t \in \mathcal{T} \quad (6)$$
$$v_t \in \{0, 1\} \qquad \forall t \in [2, |\mathcal{T}|] \quad (7)$$

Note that constraints (1) and (4) ensure that a unit cannot start up and shut down simultaneously. Since $p_t \geq 0$, (1) and (4) impose the inequalities $v_t \leq u_t$ and $u_t \leq 1 - w_t$, respectively, which combined become $v_g + w_g \leq 1$.

Although we consider the variable w_t to indicate whether the generator is shut down in period t, this variable is completely determined in terms of u_t and v_t. From (3), $w_t = v_t - u_t + u_{t-1}$. Consequently, the set of constraints (1)-(7) can be rewritten as the following polyhedron $\mathcal{P}$:

$$p_t \leq \Delta(u_t - v_t) \qquad \forall t \in [2, |\mathcal{T}|] \quad (8)$$
$$p_{t-1} \leq \Delta(u_t - v_t) \qquad \forall t \in [2, |\mathcal{T}|] \quad (9)$$
$$v_t \geq u_t - u_{t-1} \qquad \forall t \in [2, |\mathcal{T}|] \quad (10)$$
$$v_t \leq 1 - u_{t-1} \qquad \forall t \in [2, |\mathcal{T}|] \quad (11)$$
$$p_t \geq 0 \qquad \forall t \in \mathcal{T} \quad (12)$$
$$0 \leq u_t \leq 1 \qquad \forall t \in \mathcal{T} \quad (13)$$
$$0 \leq v_t \leq 1 \qquad \forall t \in [2, |\mathcal{T}|] \quad (14)$$

where $\Delta = \overline{P} - \underline{P}$.

Proposition 1. *Considering $|\mathcal{T}| = 2$, $\mathcal{P}$ is full dimensional, dim$(\mathcal{P}) = n = 5$ as $x^{(1)}, \ldots, x^{(6)}$, from Fig. 1, are six affinely independent points in $\mathcal{P}$.*

Proof. From [2, Definition 9.4], the $n + 1 = 6$ vectors $\left(x^{(1)}, 1\right)$ ..., $\left(x^{(6)}, 1\right)$ are linearly independent. That is, considering $\Delta > 0$, e.g., $\Delta = 2$, then the determinant of the matrix made by the 6 vectors is equal to 4. $\square$

Proposition 2. *The six points $x^{(1)}..., x^{(5)}, x^{(7)}$ are also affinely independent.*

Proof. The 6 vectors $\left(x^{(1)}, 1\right) ..., \left(x^{(5)}, 1\right), \left(x^{(7)}, 1\right)$ are linearly independent (the determinant of the matrix made by the 6 vectors is equal to -4 if $\Delta = 2$). $\square$

Proposition 3. *The inequality* (8) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. From [2, Definition 9.5], see facet proof in [2] Approach 1 page 144. The $n = 5$ affinely independent points $x^{(1)}..., x^{(3)}, x^{(5)}, x^{(7)}$ (see Proposition 2) satisfy the equality $p_t = \Delta\left(u_t - v_t\right)$. $\square$

Proposition 4. *The inequality* (9) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(1)}..., x^{(3)}, x^{(5)}, x^{(6)}$ (see Proposition 1) satisfy the equality $p_{t-1} = \Delta\left(u_t - v_t\right)$. $\square$

Proposition 5. *The inequality* (10) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(1)}..., x^{(2)}, x^{(4)}..., x^{(6)}$ (see Proposition 1) satisfy the equality $v_t = u_t - u_{t-1}$ $\square$

Proposition 6. *The inequality* (11) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(2)}..., x^{(6)}$ (see Proposition 1) satisfy the equality $v_t = 1 - u_{t-1}$ $\square$

Proposition 7. *The inequalities* (12) *are facet-defining of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(1)}..., x^{(4)}, x^{(6)}$ (see Proposition 1) satisfy the equality $p_t = 0$; and The $n = 5$ affinely independent points $x^{(1)}..., x^{(4)}, x^{(7)}$ (see Proposition 2) satisfy the equality $p_{t-1} = 0$. $\square$

Proposition 8. *The inequality* $v_t \geq 0$ (14) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(1)}, x^{(3)}..., x^{(6)}$ (see Proposition 1) satisfy the equality $v_t = 0$. $\square$

Proposition 9. *From* (13)-(14)*, The inequalities* $v_t \leq 1\ \forall t \in [2, |\mathcal{T}|]$ *and* $0 \leq u_t \leq 1\ \forall t \in \mathcal{T}$ *are redundant.*

Proof. The facets $v_t \geq 0$ (see Proposition 8) and $v_t \leq 1 - u_{t-1}$ (see Proposition 6) guarantee (dominate over) the inequality $u_{t-1} \leq 1$.

The facet (8) (see Proposition 3) imposes $u_t \geq v_t$ and facet $v_t \geq u_t - u_{t-1}$ (see Proposition 5) guarantee $u_{t-1} \geq 0$.

The facet $v_t \leq 1 - u_{t-1}$ (see Proposition 6) and $u_{t-1} \geq 0$ ensure $v_t \leq 1$.

The inequality $u_t \geq v_t$ and the facet $v_t \geq 0$ (see Proposition 8) ensure $u_t \geq 0$.

The facets $1 - u_{t-1} \geq v_t$ (see Proposition 6) and $v_t \geq u_t - u_{t-1}$ (see Proposition 5) guarantee $u_t \leq 1$. $\square$

Theorem 10. *Because of the previous nine prepositions, $\mathcal{P}^*$ provides the rational polyhedron of the convex hull of $\mathcal{P}$, $\mathcal{P}^* = conv(\mathcal{P})$, where $\mathcal{P}^*$ is the polyhedron $\mathcal{P}$ without the redundant inequalities presented in Proposition 9.*

$$
\begin{array}{rcccccc}
 & & u_{t-1} & u_t & v_t & p_{t-1} & p_t \\
x^{(1)} & = & (0 & 0 & 0 & 0 & 0) \\
x^{(2)} & = & (0 & 1 & 1 & 0 & 0) \\
x^{(3)} & = & (0 & 1 & 1 & 0 & \delta^U) \\
x^{(4)} & = & (1 & 0 & 0 & 0 & 0) \\
x^{(5)} & = & (1 & 0 & 0 & \delta^D & 0) \\
x^{(6)} & = & (1 & 1 & 0 & 0 & 0) \\
x^{(7)} & = & (1 & 1 & 0 & \Delta & \Delta) \\
x^{(8)} & = & (1 & 1 & 0 & \Delta & 0) \\
x^{(9)} & = & (1 & 1 & 0 & 0 & \Delta)
\end{array}
$$

Fig. 2: I think that those are all the possible vertexes

2. Full Formulation with SU and SD Capabilities

The total unit production is $\underline{P} \cdot u_t + p_t$: the minimum power output $\underline{P}$ that is generated just by being committed, and the generation over that minimum p_t. The generation limits over the power output and considering unit's startup and shutdown capabilities are set as follows Morales-España et al. [1]:

$$p_t \leq \left(\overline{P}-\underline{P}\right) u_t - \left(\overline{P}-SU\right) v_t \qquad \forall t \in \mathcal{T} \quad (15)$$

$$p_{t-1} \leq \left(\overline{P}-\underline{P}\right) u_{t-1} - \left(\overline{P}-SD\right) w_t \qquad \forall t \in [2, |\mathcal{T}|] \quad (16)$$

$$u_t - u_{t-1} = v_t - w_t \qquad \forall t \in [2, |\mathcal{T}|] \quad (17)$$

$$u_t \geq v_t \qquad t \in [2, |\mathcal{T}|] \quad (18)$$

$$w_t \leq 1 - u_t \qquad \forall t \in [2, |\mathcal{T}|] \quad (19)$$

$$p_t \geq 0 \qquad \forall t \in \mathcal{T} \quad (20)$$

$$u_t \in \{0, 1\} \qquad \forall t \in \mathcal{T} \quad (21)$$

$$v_t \in \{0, 1\} \qquad \forall t \in [2, |\mathcal{T}|] \quad (22)$$

From (3), $w_t = v_t - u_t + u_{t-1}$. Consequently, the set of constraints (1)-(7) can be rewritten as the following polyhedron $\mathcal{P}$:

$$p_t \leq \Delta u_t - \delta^U v_t \qquad \forall t \in \mathcal{T} \quad (23)$$

$$p_{t-1} \leq SD \cdot u_{t-1} - \delta^D \left(v_t - u_t\right) \qquad \forall t \in [2, |\mathcal{T}|] \quad (24)$$

$$v_t \geq u_t + u_{t-1} \qquad \forall t \in [2, |\mathcal{T}|] \quad (25)$$

$$u_t \geq v_t \qquad t \in [2, |\mathcal{T}|] \quad (26)$$

$$v_t \leq 1 - u_{t-1} \qquad \forall t \in [2, |\mathcal{T}|] \quad (27)$$

$$p_t \geq 0 \qquad \forall t \in \mathcal{T} \quad (28)$$

$$0 \leq v_t \leq 1 \in \{0, 1\} \qquad \forall t \in [2, |\mathcal{T}|] \quad (29)$$

$$0 \leq u_t \leq 1 \in \{0, 1\} \qquad \forall t \in \mathcal{T} \quad (30)$$

where $\Delta = \overline{P}-\underline{P}$, $\delta^U = \overline{P}-SU$, and $\delta^D = \overline{P}-SD$.

Proposition 11. *Considering $|\mathcal{T}| = 2$, $\mathcal{P}$ is full dimensional, dim$(\mathcal{P}) = n = 5$ as $x^{(1)}, \ldots, x^{(6)}$, from Fig. 2, are six affinely independent points in $\mathcal{P}$.*

Proof. From [2, Definition 9.4], the $n + 1 = 6$ vectors $\left(x^{(1)}, 1\right)$..., $\left(x^{(6)}, 1\right)$ are linearly independent. That is, considering, for example, $\Delta = 2$, $\delta^U = 4/3$, $\delta^D = 2/3$, then the determinant of the matrix made by the 6 vectors is equal to -0.8889. $\square$

Proposition 12. *The inequality* (23) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(1)}, x^{(3)}..., x^{(5)}, x^{(9)}$ satisfy the equality $p_t = \Delta u_t - \delta^U v_t$. That is, considering, for example, $\Delta = 2$, $\delta^U = 4/3$, $\delta^D = 2/3$, then the determinant of the matrix made by the 6 vectors (including $x^{(8)}$) is equal to 1.333. ☐

Proposition 13. *The inequality* (24) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(1)}..., x^{(3)}, x^{(5)}, x^{(8)}$ satisfy the equality $p_{t-1} = SD \cdot u_{t-1} - \delta^D (v_t - u_t)$. That is, considering, for example, $\Delta = 2$, $\delta^U = 4/3$, $\delta^D = 2/3$, then the determinant of the matrix made by the 6 vectors (including $x^{(9)}$) is equal to -2.6667. ☐

Proposition 14. *The inequality* (25) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(1)}..., x^{(3)}, x^{(8)}, x^{(9)}$ satisfy the equality $v_t = u_t + u_{t-1}$. That is, considering, for example, $\Delta = 2$, $\delta^U = 4/3$, $\delta^D = 2/3$, then the determinant of the matrix made by the 6 vectors (including $x^{(5)}$) is equal to -2.6667. ☐

Proposition 15. *The inequality* (26) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(1)}..., x^{(5)}$ satisfy the equality $u_t = v_t$. That is, considering, for example, $\Delta = 2$, $\delta^U = 4/3$, $\delta^D = 2/3$, then the determinant of the matrix made by the 6 vectors (including $x^{(6)}$) is equal to -0.8889. ☐

Proposition 16. *The inequality* (27) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(2)}..., x^{(6)}$ satisfy the equality $v_t = 1 - u_{t-1}$. That is, considering, for example, $\Delta = 2$, $\delta^U = 4/3$, $\delta^D = 2/3$, then the determinant of the matrix made by the 6 vectors (including $x^{(1)}$) is equal to -0.8889. ☐

Proposition 17. *The inequality* (28) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(1)}..., x^{(4)}, x^{(6)}$ satisfy the equality $p_{t-1} = 0$. The $n = 5$ affinely independent points $x^{(1)}, x^{(2)}, x^{(4)}..., x^{(6)}$ satisfy the equality $p_t = 0$. That is, considering, for example, $\Delta = 2$, $SU = 4/3$, $SD = 2/3$, then the determinant of the matrix made by the 6 vectors $\left(x^{(1)}, 1\right)..., \left(x^{(6)}, 1\right)$ is equal to -0.8889. ☐

Proposition 18. *The inequality* (29) *defines a facet of conv*$(\mathcal{P})$*, for* $|\mathcal{T}| = 2$.

Proof. The $n = 5$ affinely independent points $x^{(1)}, x^{(5)}, x^{(6)}, x^{(8)}, x^{(9)}$ satisfy the equality $v_t = 0$. That is, considering, for example, $\Delta = 2$, $\delta^U = 4/3$, $\delta^D = 2/3$, then the determinant of the matrix made by the 6 vectors (including $x^{(3)}$) is equal to -4. ☐

Proposition 19. *From* (23)-(29)*, The inequalities* $v_t \leq 1\ \forall t \in [2, |\mathcal{T}|]$ *and* $0 \leq u_t \leq 1\ \forall t \in \mathcal{T}$ *are redundant.*

Proof. See Proof of Proposition 9. ☐

Theorem 20. *Because of the previous nine prepositions,* $\mathcal{P}^*$ *provides the rational polyhedron of the convex hull of* $\mathcal{P}$*,* $\mathcal{P}^* = conv(\mathcal{P})$*, where* $\mathcal{P}^*$ *is the polyhedron* $\mathcal{P}$ *without the redundant inequalities presented in Proposition 19.*

References

[1] Morales-España, G., Latorre, J. M., Ramos, A., 2012. Tight and compact MILP formulation for the thermal unit commitment problem. IEEE Transactions on Power SystemsAccepted for publication (DOI: 10.1109/TPWRS.2013.2251373), online preprint.
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[2] Wolsey, L., 1998. Integer Programming. Wiley-Interscience.