BENDERS DECOMPOSITION FOR MIXED-INTEGER HYDROTHERMAL PROBLEMS BY LAGRANGEAN RELAXATION

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Abstract – Decomposition models with integer variables usually decompose into a master problem that comprises all the integer variables and subproblems, which evaluate the remaining variables. Subproblems with integer variables introduce additional difficulties and require the use of nonlinear duality theory. In this paper we address the solution of a mixed integer hydrothermal coordination problem combining nested Benders decomposition and lagrangean relaxation. An extensive computational study applied to a large-scale hydrothermal problem is presented.

***Keywords:** Nested Benders decomposition, lagrangean relaxation, stochastic optimization.*

1 INTRODUCTION

Hydrothermal coordination problems with multiple stages usually present staircase structures that can be exploited to decompose the problem [8]. In this sense, Benders [1] decomposition appears as a natural way to solve LP problems, breaking up the hydro reserve balance equations that connect stages.

This situation is generalized when the problem is divided into multiple stages via nested decomposition and linear stochastic problems are solved forming adequate subproblems. The common feature of this decomposition technique is the convexity requirement for the subproblems, which provides an easy way of calculating dual variables with a general algorithm. This requirement limits the appearance of integer variables in the model to those formulated at the master problem and simplifies further model stages because of the necessity of considering only continuous variables at subproblems.

However, models can strictly require the use of integer variables at subproblems and specific decomposition algorithms have to be developed. This is the situation of the model presented in this paper, in which integer variables appear when modeling commitment decisions for thermal units and input-output nonlinear curves for hydroelectric units. These curves are approximated by piecewise linear functions and binary variables are used to select a specific segment over the polygonal.

Under the Benders framework, when subproblems are convex (e.g., linear) the recourse function is also convex. This provides the possibility of solving the problem by an outer approximation algorithm of this recourse function, together with an infeasibility criterion that excludes infeasible first stage solutions by introducing feasibility cuts.

The situation is slightly different for nonconvex subproblems because the recourse function is nonconvex in general. Introducing lagrangean relaxation to solve those subproblems approximates the lower convex envelope of the recourse function, and is a natural way to solve decomposition problems with integer variables at all stages.

This approach is easily extended to nested situations and stochastic problems formulated via a scenario tree. The presented algorithm divides both a deterministic and a stochastic problem into subproblems and sequentially solves them passing down variable decisions and receiving back dual variables obtained by lagrangean relaxation to form outer approximations of the recourse functions.

Tree traversing strategies turn out to be quite different when looking for an optimal solution or just looking for a feasible one. In this sense, different strategies have been developed and tested and results are presented regarding execution times and quality of the integer solutions. The stochastic problem allows the aggregation of scenario tree nodes forming arbitrary subtrees that are solved with this technique as the algorithm proceeds. Numerical results of different aggregation techniques are also reported.

The algorithm presented in this paper is an extension of decomposition theory and results from Benders [1], Geoffrion [5,6], Van Slyke and Wets [9] in stochastic programming and convex analysis. It deals with infeasibilities at LP and MIP subproblems and produce optimality cuts at feasible LP and MIP subproblems. It is especially suitable for very large-scale problems due to the possibility of solving much bigger problems than those accepted by current MIP solvers.

An alternative approach to solve MIP problems within a decomposition framework is to consider the *integer L-shaped method* [7]. It has the disadvantage of incorporating extra discrete variables as the algorithm proceeds. On the contrary, it does not requite lagrangean relaxation and could also reduce time solutions.

The paper presents a review of Benders decomposition and continues with lagrangean relaxation and its inclusion within a decomposition framework as classical theory does. Remaining sections are devoted to numerical implementation and results. Conclusion states limitations, advantages and future improvements of present work.

2 BENDERS DECOMPOSITION

We consider problems with the following staircase structure

$$(P)\quad \begin{aligned} &\min z = c^1 x + c^2 y \\ &A^{11}x \qquad\quad = b^1 \\ &A^{21}x + A^{22}y = b^2 \\ &x \ge 0, y \ge 0; x, y \in \mathbb{R}^n \end{aligned} \qquad (1)$$

Benders decomposition turns the previous problem into

$$(MP)\quad \begin{aligned} &\min z = c^1 x + \theta(x) \\ &A^{11}x = b^1 \\ &x \ge 0, x \in V \end{aligned} \qquad (2)$$

where

$$(S)\ \theta(x) = \left\{\min c^2 y \,/\, A^{22}y = b^2 - A^{21}x \,/\, y \ge 0\right\} \qquad (3)$$

and V represents a set that guarantees second stage feasibility for first stage solutions. For LP problems strong linear duality leads to convexity of the recourse function because

$$\theta(x) = \left\{\max \pi(b^2 - A^{21}x), \pi A^{22} \le c^2\right\} \qquad (4)$$

and convexity for the feasibility first stage set V results in a similar way by a direct use of Farkas' lemma.

$$V = \left\{x \in \mathbb{R}^n \,/\, \sigma(b^2 - A^{21}x) \le 0 \quad \forall \sigma \,/\, \sigma A^{22} \le 0\right\} \qquad (5)$$

The difficulty is that the set V and the function θ are only known by their implicit definitions. So a natural way to proceed consists on an outer approximation algorithm that solves a relaxed master problem *RMP* and introduces additional constraints when necessary by solving the updated subproblem *S* and obtaining an optimality cut or a feasibility cut.

$$(RMP)\quad \begin{aligned} &\min z = c^1 x + \theta \\ &A^{11}x = b^1 \\ &\sigma^k A^{21}x \ge \sigma^k b^2 \qquad k = 1,\ldots,K \\ &\theta \ge \pi^j(b^2 - A^{21}x) \quad j = 1,\ldots,J \\ &x \ge 0 \end{aligned} \qquad (6)$$

where $\sigma^k \in \operatorname{extr}\left\{\sigma A^{22} \le 0, -e \le \sigma \le -e\right\}$ and $\pi^j \in \operatorname{extr}\left\{\pi A^{22} \le c^2\right\}$.

The *two-stage linear Benders decomposition algorithm* is summarized in the following steps

step 0 define $\mathrm{tol} > 0$, $\underline{z} = -\infty$, $\overline{z} = \infty$
step 1 solve relaxed master problem *RMP*
obtain (x^n, θ^n)
obtain lower bound $\underline{z} = c^1 x^n + \theta^n$
step 2 form RHS subproblem
solve subproblem *S* and obtain y^n
if subproblem is infeasible
obtain σ^{k+1}; do $K = K \cup \{k+1\}$
if subproblem is feasible
obtain π^{j+1}; do $J = J \cup \{j+1\}$
obtain upper bound $\overline{z} = c^1 x^n + c^2 y^n$
step 3 (stopping rule)
if $\overline{z} - \underline{z} < \overline{z} \cdot \mathrm{tol}$ then stop
goto step 1

Benders decomposition algorithm is generalized to multistage and stochastic situations forming the equivalent deterministic problem and solving it via nested decomposition.

3 LAGRANGEAN RELAXATION

The assumption of continuous variables and linear subproblems is dropped away when other characteristics have to be modeled and binary or discrete variables are required. This situation notably increases problem difficulty because branch and bound or cutting plane methods are necessary to solve the problem and sensitivity analysis is no longer valid.

The general structure of the problem remains equal and its staircase structure invites to proceed with an outer approximation algorithm in a similar way to the linear case.

$$(P)\quad \begin{aligned} &\min z = c^1 x + c^2 y \\ &A^{11}x \qquad\quad = b^1 \\ &A^{21}x + A^{22}y = b^2 \\ &\qquad\quad A^{32}y = b^3 \\ &x \ge 0, y \ge 0, x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{m_1}, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2} \end{aligned} \qquad (7)$$

Subproblem can be interpreted as a parametrized family of subproblems and the recourse function $\theta(x)$ as the perturbation function that gives the optimal value of the subproblem for each first stage value.

$$(S)\ \theta(x) = \left\{\begin{aligned} &\min c^2 y \,/\, A^{22}y = b^2 - A^{21}x, \\ &A^{32}y = b^3, y \ge 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2} \end{aligned}\right\} \qquad (8)$$

This perturbation function is nonconvex and the solution of the master problem would require its convexification. Geoffrion's results [5] establish that the lower convex envelope of the perturbation function is precisely the function

$$\begin{aligned} &\theta^*(x) = \min c^2 y \\ &A^{22}y = b^2 - A^{21}x, \\ &y \in \operatorname{conv}\left\{A^{32}y = b^3, y \ge 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\} \end{aligned} \qquad (9)$$

It should be taken into account that this convexification is computed over $\mathbb{R}^n$ and that, in case x is constrained to belong to $X = \left\{A^{11}x = b^1\right\}$, the previous function represents a lower approximation to the lower convex envelope. However, it is enough to obtain good first stage variable solutions. The difference between this approximation and the exact lower convex envelope can be interpreted as an extension of the *duality gap.*

Trivially, the previous representation satisfies the integrality property assumption that guarantees that the value of the previous problem is not altered by solving it via lagrangean relaxation. So the lagrangean is formed as

$$L^*(y,\lambda) = c^2 y + \lambda\left(A^{22}y - b^2 + A^{21}x\right) \tag{10}$$

and the dual function

$$(S^*)\quad \begin{array}{l} w^*(\lambda) = \min c^2 y + \lambda\left(A^{22}y - b^2 + A^{21}x\right) \\ y \in \operatorname{conv}\left\{A^{32}y = b^3, y \geq 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\} \end{array} \tag{11}$$

The dual problem D^* is defined as

$$(D^*)\ \max\left\{w(\lambda), \lambda \geq 0\right\} \tag{12}$$

Realize that the dual function is concave and it can be obtained as the pointwise minimum of a family of linear functions. It is traditionally optimized by a cutting plane method forming a relaxed master dual problem RD^* that is updated with different solutions of the dual subproblem S^*.

$$(RD^*)\quad \begin{array}{l} \max w \\ w \leq c^2 y^k + \lambda\left(A^{22}y^k - b^2 + A^{21}x\right) \\ y^k \in \operatorname{conv}\left\{A^{32}y = b^3, y \geq 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\} \\ k: 1,\dots,K \end{array} \tag{13}$$

The *lagrangean relaxation algorithm* is stated as

step 0 define $\text{tol} > 0$, $\underline{z} = -\infty$, $\overline{z} = \infty$
step 1 solve relaxed master dual problem *RD**
obtain dual values (w^k, λ^k)
obtain upper bound $\overline{z} = w^k$
step 2 form objective function subproblem *S**
solve subproblem *S** and obtain y^{k+1}
obtain lower bound $\underline{z} = c^2 y^{k+1}$
do $K = K \cup \{k+1\}$
step 3 (stopping rule)
if $\overline{z} - \underline{z} < \overline{z} \cdot \text{tol}$ then stop
goto step 1

3.1 Infeasibility

In previous developments we assumed second stage subproblem feasibility. This condition should be tested prior to solving the problem, in an equivalent manner as phase 1 proceeds within the simplex algorithm. At this moment we can assume without loosing generality that the system

$$Y' = \operatorname{conv}\left\{A^{32}y = b^3, y \geq 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\} \tag{14}$$

has a solution. This assumption does not represent any extra condition for the problem because infeasibility of this problem implies infeasibility of the original problem, which is assumed to be feasible.

Farkas' lemma translates the concept of infeasibility to the solution of an optimization problem that represents the minimization of infeasibilities. In case of positive solution of this problem then the dual variables are used to formulate a constraint that excludes that infeasible solution. This idea is easily extended to the working case and a minimization problem is formed in the following way

$$\begin{array}{l} \min v^+ + v^- \\ A^{22}y + Iv^+ - Iv^- \leq b^2 - A^{21}x \\ y \in \operatorname{conv}\left\{A^{32}y = b^3, y \geq 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\} \end{array} \tag{15}$$

which has a suitable structure to be solved via lagrangean relaxation.

We consider the lagrangean

$$L_*(y,\lambda) = v^+ + v^- + \lambda\left(A^{22}y + Iv^+ - Iv^- - b^2 + A^{21}x\right) \tag{16}$$

and the dual function

$$\begin{array}{l} w_*(\lambda) = \min v^+ + v^- + \lambda\left(A^{22}y + Iv^+ - Iv^- - b^2 + A^{21}x\right) \\ y \in \operatorname{conv}\left\{A^{32}y = b^3, y \geq 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\} \\ v^+ \geq 0, v^- \geq 0 \end{array} \tag{17}$$

or

$$\begin{array}{l} w_*(\lambda) = \min (1+\lambda)v^+ + (1-\lambda)v^- + \lambda(A^{22}y - b^2 + A^{21}x) \\ y \in \operatorname{conv}\left\{A^{32}y = b^3, y \geq 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\} \\ v^+ \geq 0, v^- \geq 0 \end{array}$$

(18)

which turns the dual function into

$$\begin{array}{l} w_*(\lambda) \equiv -\infty \ \text{ if } \lambda \notin [-1,1] \\ w_*(\lambda) = \min \lambda\left(A^{22}y - b^2 + A^{21}x\right) \\ y \in \operatorname{conv}\left\{A^{32}y = b^3, y \geq 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\} \end{array} \tag{19}$$

Then, the dual problem D_* consists on

$$(D_*)\ \max\left\{w_*(\lambda), -1 \leq \lambda \leq 1\right\} \tag{20}$$

and it is solved with an outer approximation algorithm in an equivalent way as the feasibility problem is solved. Both situations are solved together in an algorithm that gives back correct dual variables and objective functions for feasible and infeasible situations. It takes advantage of values y, that form outer approximations of the dual function for the feasibility and the infeasibility case, by considering the following generalized subproblem objective function and the generalized constraint for the relaxed dual problems

$$(1-\alpha)c^2 y + \lambda\left(A^{22}y - b^2 + A^{21}x\right) \tag{21}$$

$$w \leq (1-\alpha)c^2 y^k + \lambda\left(A^{22}y^k - b^2 + A^{21}x\right) \tag{22}$$

and setting $\alpha = 1$ for phase 1 (feasibility) and $\alpha = 0$ for phase 2 (optimality). So the relaxed dual problem *RD* and subproblem *S* are defined as

$$(RD)\quad \begin{array}{l} \max w \\ w \leq (1-\alpha)c^2 y^k + \lambda\left(A^{22}y^k - b^2 + A^{21}x\right) \\ y^k \in \operatorname{conv}\left\{A^{32}y = b^3, y \geq 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\} \\ k: 1,\dots,K \end{array} \tag{23}$$

$$(S)\quad \begin{array}{l} w(\lambda) = \min (1-\alpha)c^2 y + \lambda\left(A^{22}y - b^2 + A^{21}x\right) \\ y \in \operatorname{conv}\left\{A^{32}y = b^3, y \geq 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}\right\} \end{array} \tag{24}$$

The *lagrangean relaxation algorithm with phase 1 that minimizes infeasibilities* is then summarized.

step 0 define $\mathrm{tol} > 0$, $\underline{z} = -\infty$, $\overline{z} = \infty$, $\alpha = 1$
step 1 solve relaxed master dual problem *RMD*
obtain dual values (w^k, λ^k)
obtain upper bound $\overline{z} = w^k$
step 2 form objective function subproblem *S*
solve subproblem *S* and obtain y^{k+1}
obtain lower bound $\underline{z} = c^2 y^{k+1}$
do $K = K \cup \{k+1\}$
step 3 (switching and stopping rule)
if $\overline{z} - \underline{z} < \overline{z} \cdot \mathrm{tol}$ and $\alpha = 1$
$\alpha = 0$
$\underline{z} = -\infty$, $\overline{z} = \infty$
goto step 1
if $\overline{z} - \underline{z} < \overline{z} \cdot \mathrm{tol}$ and $\alpha = 0$ stop

This algorithm provides correct values to form outer approximations of the recourse function for a Benders decomposition situation when the recourse function is nonconvex. Within Benders decomposition, the linear solution of the subproblem is replaced by its solution via lagrangean relaxation, where the relaxed equations are the hydro reserve balance equations that connect stages and create the RHS perturbation function.

4 NESTED DECOMPOSITION

Nested linear decomposition arises when a second-stage subproblem is also solved via decomposition techniques, thus forming a chain of consecutive subproblems that are solved in an algorithm. Solutions of the ancestor subproblem modify RHS values for the current subproblem, while this gives back dual variables that form outer approximation of the convex recourse functions. Traversing strategies at this moment become crucial for reaching a solution in a reasonable time.

The use of traversing strategies for a MIP problem implies that a problem has to be solved using lagrangean relaxation when a dual variable is required to obtain an outer approximation for its ancestor problem.

Sequential solutions of relaxed subproblems lead to inappropriate primal solutions used to form outer approximations of the dual function and have to be eliminated when this is detected.

To avoid unnecessary complicating notation just consider a three stage problem

$$(P)\quad \begin{aligned} &\min c^1x + c^2y + c^3z \\ &A^{11}x = b^1 \\ &A^{21}x + A^{22}\mathrm{y} = b^2 \\ &\qquad \mathrm{A}^{23}y + A^{33}z = b^3 \\ &x, y, z \ge 0 \\ &x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{m_1}, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2}, z \in \mathbb{R}^{n_3} \times \mathbb{Z}^{m_3} \end{aligned} \tag{25}$$

and its solution via a three stage decomposition scheme.

In this situation the second stage recourse function is relaxed and replaced by a partial outer approximation and the problem is solved by branch and bound to obtain a primal feasible solution for the third stage. The expression is generalized to deal simultaneously with feasibility and optimality cuts.

$$(MP)\quad \begin{aligned} &\min c^2y + \theta \\ &A^{22}y = b^2 - A^{21}x \\ &(1-\alpha_3^j)\theta \ge \pi^j(b^3 - A^{32}y) \quad j:1,\ldots,J \\ &y \ge 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2} \end{aligned} \tag{26}$$

where

$$\pi^j \in \operatorname{extr}\{\pi A^{33} \le c^2\} \cup \operatorname{extr}\{\sigma A^{33} \le 0, -e \le \sigma \le -e\}$$

However, when a correct dual variable is necessary to form a cutting plane for the ancestor problem, then the previous problem ought to be solved via lagrangean relaxation, maximizing an outer approximation of the dual function given by

$$(RD)\quad \begin{aligned} &\max w \\ &w \le (1-\alpha_2)c^2y^q + \theta^q + \lambda\left(A^{22}y^q - b^2 + A^{21}x\right) \\ &q:1,\ldots,Q \end{aligned} \tag{27}$$

where

$$(y^q, \theta^q) \in \begin{Bmatrix} (1-\alpha^j)\theta \ge \pi^j(b^3 - A^{32}y) \quad j:1,\ldots,J \\ y \ge 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2} \end{Bmatrix} \tag{28}$$

The dual function takes the form

$$(S)\quad \begin{aligned} &w(\lambda) = \min(1-\alpha_2)(c^2y + \theta) + \lambda(A^{22}y - b^2 + A^{21}x) \\ &(1-\alpha_3^j)\theta \ge \pi^j(b^3 - A^{32}y) \quad j:1,\ldots,J \\ &y \ge 0, y \in \mathbb{R}^{n_2} \times \mathbb{Z}^{m_2} \end{aligned} \tag{29}$$

Extra planes at second stage subproblem eliminate primal variables obtained at previous iterations of the decomposition algorithm in case these variables do not satisfy the latest equation. These points have to be removed from the pool of points that forms the dual master problem.

Consider a new cut created via a third-stage problem solution

$$(1-\alpha^{j+1})\theta \ge \pi^{j+1}(b^3 - A^{32}y) \tag{30}$$

and (y^q, θ^q) such that

$$(1-\alpha^{j+1})\theta^q < \pi^{j+1}(b^3 - A^{32}y^q) \tag{31}$$

then the point (y^q, θ^q) has to be removed from the pool of points.

Traversing strategies take profit of a lot of possible combinations to obtain primal or dual variables. Among all of them fastpass strategy [8] outstands as an efficient one. It consists on solving the subproblems from the first to the last and then solving the subproblems from the last to the first obtaining dual variables. For MIP problems a fastpass strategy remains almost identical, taking into account that lagrangean relaxation is necessary to form correct dual variables.

Thus, lagrangean relaxation has been introduced within a Benders decomposition framework when dual variables are requested from MIP subproblems. Finding infeasibilities at middle stages when a forward pass is performed commands the algorithm to bounce and to start the backward pass solving all problems with lagrangean relaxation.

A *multistage nested Benders decomposition algorithm for MIP problems* is summarized in the following steps

step 0 define $\mathrm{tol} > 0$, $\underline{z} = -\infty$, $\overline{z} = \infty$
step 1 (forward pass)
$T = P$
for $p = 1$ to $p = P - 1$
form RHS problem
solve *MP* problem using branch and bound
if problem infeasible
$T = p$
goto step 3
if problem feasible obtain x^p
if $p = 1$ obtain lower bound $\underline{z} = c^1 x + \theta^1$
Step 2 (backward pass)
for $p = T$ to $p = 2$
update problem with new cut ($p \neq P$)
update pool of points Q_p to use
lagrangean relaxation ($p \neq P$)
solve problem MP using lagrangean relaxation
obtain dual variable π_p^j
obtain feasibility indicator α_p^j
if $p = P$
obtain upper bound $\overline{z} = \sum_{p=1}^{P-1} c^p x^p + w^P$
Step 3 (stopping rule)
If $\overline{z} - \underline{z} < \overline{z} \cdot \mathrm{tol}$ then stop

where $p: 1, ..., P$ are the problem stages, x^p are the variables of the primal problems, c^p are the objective function coefficients, α_p^j represents the feasibility indicator, π_p^j denotes dual variables, Q_p is the pool of points at period p and w_p is the dual function at period p.

5 STOCHASTIC DECOMPOSITION

Stochastic problems arise when random parameters appear all over the model and optimum decisions are transformed into random variables. Usually, stochastic problems present a first deterministic stage and remaining stages where random parameters create multiple possibilities that are usually represented alongside a scenario tree, see figure 1.

These problems maintain a matrix structure suitable to be solved using decomposition techniques. Stochasticity is introduced in our problem when considering inflows as random parameters. A natural way to solve these problems is to consider an equivalent deterministic problem and solve it by generalized Benders decomposition approach.

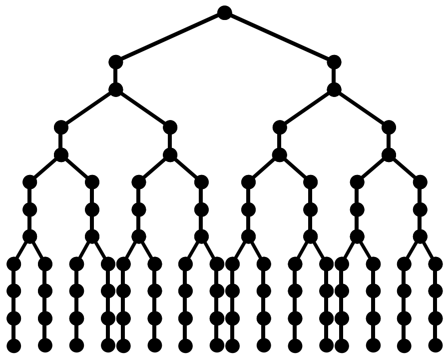

Figure 1. Scenario tree.

Among these generalizations, both monocut and multicut approaches [2] create a Benders cut once all descendent subproblems have been solved. We choose a multicut approach because it gives the possibility of creating a feasibility and an optimality cut independently, based on descendent subproblems' states while a monocut approach forms a unique cut weighting up cuts. A monocut approach cannot deal with different descendent subproblem states.

Multistage stochastic LP problems are solved in [4] aggregating the nodes of the tree forming arbitrary subtrees, as in the figure 2, which will be solved during the algorithm. Numerical results recommend forming bigger subtrees instead of smaller ones in order to speed up time solutions. This is due to the time requested by the algebraic modeling language to create a problem and communication time with commercial solvers.

Benders' decomposition for mixed hydrothermal coordination problems is immediately extended to this situation of subtree decomposition. This flexibility gives the possibility of performing a great number of numerical tests about time and quality of the solutions. Extensive numerical results are reported on section 9.

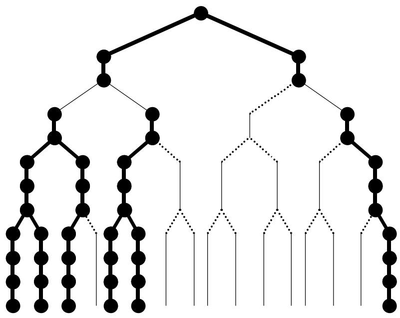

Figure 2. Scenario tree divided into subtrees.

6 TRAVERSING STRATEGIES

One drawback of lagrangean relaxation is the great number of iterations required to reach an optimum or near-optimum multiplier. However, as we indicate later, a minor number of iterations are needed to identify an infeasible problem. This leads to consider traversing strategies that look for a feasible solution instead of looking for an optimum one. This section briefly describes natural traversing strategies suitable for stochastic problems as generalizations of the deterministic problem ones.

6.1 Fastpass traversing strategy for LP problems

This strategy consists on traversing the scenario tree downwards and solving subproblems associated with those subtrees relaxing integrality conditions. Then traversing the tree upwards creating cutting planes for the recourse functions. Once a subtree is identified to be infeasible at a downward pass, the algorithm bounces to start backward pass and create feasibility cuts that eliminate previous stage solutions.

6.2 Fastpass traversing strategy for MIP problems

This strategy generalizes the linear strategy solving subproblems downwards with branch and bound techniques and via lagrangean relaxation at backward pass. It also bounces when branch and bound identifies a subtree to be infeasible and a backward pass is started solving infeasible subproblems via lagrangean relaxation. This strategy directly generalizes the nested strategy for deterministic problems presented in the last algorithm.

6.3 Feasibility traversing strategy for MIP problems

When a subproblem is identified to be infeasible previous algorithm strategies bounce and start a backward pass creating optimality or feasibility cuts up to the root node subtree. Optimality cuts are computationally expensive to calculate and it is a natural procedure to perform an algorithm creating only feasibility cuts. This is done by solving subproblems at a backward pass by branch and bound and bouncing to start downward pass if these are detected to be feasible. In other case the subproblems are solved by lagrangean relaxation to create a suitable feasibility cut. This strategy has the drawback of solving a problem twice.

These strategies are quite similar and it is worthless to summarize all of them schematically. We limit to represent the *feasibility traversing strategy* on the next steps.

step 0 define $\text{tol} > 0$, $\underline{z} = -\infty$, $\overline{z} = \infty$, $Z = 0$, $N = (p1, sc1)$
solve problem using fastpass strategy for LP problems
step 1 (forward pass)
for $p = Z + 1$ to $p = P$
for $(p, sc) \in N$ root node
form RHS subproblem
solve subproblem using B&B
if subproblem feasible
update node pool with descendent nodes
obtain x^p
if $p = 1$
obtain lower bound $\underline{z} = c^1 x^1 + \theta^1$
if subproblem infeasible
solve using lagrangean relaxation
obtain dual variable π_p^j
obtain feasibility indicator $\alpha_p^j = 1$
update node pool with ancestor node

check feasibility of problems at same period
if at least one subproblem (p, sc) is infeasible
do $T = p$
goto step 3
if every problem is feasible
stop if $p = P$

/ deactivate nodes at level p (p, sc) /
step 2 (backward pass)
for $p = T - 1$ to $p = 1$
for $(p, sc) \in N$ root node
update problem with infeasibility cuts
solve problem using B&B
if subproblem is infeasible
solve using lagrangean relaxation
obtain dual variable π_p^j
obtain feasibility indicator $\alpha_p^j = 1$
update node pool with ancestor node
if subproblem is feasible
update node pool with descendent nodes
obtain x^p

check feasibility of problems at same period
if every problem is feasible
do $Z = p$
goto step 2

where $N = \{(p, sc)\}$ active tree nodes.

7 MODELING ISSUES

The objective problem can be either a cost minimization or profit maximization depending of the model goal. The constraints represent the operation of the thermal and hydro subsystems. The reserve-balance equations connect consecutive periods. A period will approximately represent a month composed by a pattern weekday followed by a weekend day. The variables are the operation decisions of hydro and thermal units. Integer variables appear when modeling commitment decisions of thermal units and hydroelectric units' input-output nonlinear curves. Stochasticity appears in hydro inflows.

7.1 Commitment decisions

These decisions are modeled with binary variables that indicate time periods where thermal units are operative

$$O_p \le \delta_p \overline{O}_p \tag{32}$$

where $\overline{O}_p$ is the maximum output at period p, $\delta_p = 1$ if thermal unit is operative or $\delta_p = 0$ in any other case.

7.2 Reserve balance equations

$$Rsv_{p-1} - O_p + I_p = Rsv_p \tag{33}$$

where Rsv_p reserve at the end of period p, O_p output during period p and I_p water inflows for period p.

7.3 Hydroelectric unit's input-output curves

For a hydroelectric unit the production depends both on the reserve levels Rsv of the upper and lower hydro reservoirs and on the water discharge d. So production is represented as a nonlinear function of two variables

$$O = g(d, Rsv) \tag{34}$$

This is modeled by defining a grid of values [10] for upper and lower reserve at each hydro unit and approximating the input-output function by means of the following relations.

A λ-form approach has been considered to model binary variables that approximate the input-output curves. Future research will focus on modeling these curves using a δ-form and performing numerical comparisons.

Let the grid points be $(Rsv_{up}^{s}, Rsv_{down}^{t})$, $s:1,...,S$, $t:1,...,T$ and λ_{st} the corresponding weights.

Then

$$
\begin{aligned}
&(Rsv_{up}, Rsv_{down}) = \sum_{(s,t)} \lambda_{st} (Rsv_{up}^{s}, Rsv_{down}^{t}) \\
&Power = \sum_{(s,t)} \lambda_{st} g(d, Rsv_{up}^{s}, Rsv_{down}^{t}) \qquad (35) \\
&\sum_{(s,t)} \lambda_{st} = 1
\end{aligned}
$$

where the condition that at most four neighboring points can be non zero should be added. This condition is modeled introducing extra variables α_s y β_t

$$\alpha_s = \sum_t \lambda_{st} \beta_t = \sum_s \lambda_{st} \qquad (36)$$

and commanding this variables to be SOS2 variables.

8 CASE STUDY AND MODEL IMPLEMENTATION

The case study represents the medium term operation of the Spanish electric power system. Time scope is divided into 12 periods of one month each. All thermal units are modeled individually. Hydro units are grouped by basins.

The optimization model and all the decomposition algorithms have been implemented in GAMS [3] with CPLEX 7.0 as the MIP solver.

9 NUMERICAL RESULTS

All tests have been performed over a midterm hydrothermal coordination model. We present numerical results about deterministic situations and follow with the stochastic situation. Table 1 shows results about a 12 period deterministic problem solved by a two-stage decomposition algorithm. It has been solved formulating different sized master and subproblems and different traversing strategies.

First column indicates time periods aggregated to form the master problem. Problems' characteristics are described on next columns, rows (r), variables (v), non-zeros (n) and binary variables (b). Later we indicate the number of Benders iterations required and lagrangean relaxation iterations required for the first problems solved by lagrangean relaxation. Finally we inform about execution and solution times. Execution time refers to solver time and solution time reports modeling time too. Table 1 reports solution when a MIP fastpass strategy is used. The model has been run in a PC with 256 MB of RAM memory and 550 MHz clock frequency

Table 2 presents time comparisons between fastpass linear traversing strategy and feasibility traversing strategy, focusing on time solutions and quality of the optimum. It is done for the deterministic problem using nested decomposition for different node aggregation situations. First column indicates root node of the different subproblems, so that 1, 4, 6 indicates a three stage situation where master problem has periods number 1, 2 and 3, second stage periods 4 and 5 and third stage the remaining periods. Next columns report Benders iterations for both strategies and lagrangean relaxation iterations. Finally the quality of the solution is evaluated by means of a final stopping tolerance.

	Fastpass MIP strategy			Feasibility strategy		
	Bd It	Time (s)	Quality	Bd It	Time (s)	Quality
1 2	11	3365	0			
1 3	22	4646	0.1e-6	4	59	3.2e-6
1 4	18	2908	0	5	62	3.2e-6
1 5	40	5209	0.04e-6	4	62	158.1e-6
1 6	42	2861	0.08e-6	5	55	158.1e-6
1 7	4	1075	0	9	75	2.3e-6
1 8	5	1063	0	8	88	371.5e-6
1 9	71	5426	0	13	122	229.7e-6
1 10	94	2771	0.29e-6	36	324	2.3e-6

Table 2. Fastpass versus feasibility strategy.

Time solutions at table 2 are small because subproblems turn out to be feasible and lagrangean iterations are avoided in order to produce a feasibility cut.

Master per.	Master size (r,v,n,b)				Subproblem size (r,v,n,b)				Bd Iter.	LR Iter.	Exec. time (s)	Solution time (s)
1	1605	2183	6767	146	17127	23374	72167	1606	11	166	3365	7438
1,2	3128	4265	13165	292	15604	21292	65769	1460	22	196	4646	10065
1,2,3	4644	6342	19569	438	14089	19215	59391	1314	18	176	2908	7184
1,2,3,4	6215	8489	26179	584	12517	17068	52755	1168	40	266	5209	15264
1,2,3,4,5	7714	10541	32476	730	11018	15016	46458	1022	42	243	2861	8327
1,2,3,4,5,6	9316	12721	39238	876	9416	12836	39696	876	4	266	1075	3673
1,...........,7	10906	14890	45950	1022	7826	10667	32984	730	5	304	1063	3860
1,............8	12463	17019	52521	1168	6269	8538	26413	584	71	413	5426	32176
1,............9	15598	21306	65747	1314	3134	4251	13187	438	94	336	2771	22737

Table 1. Iterations for a MIP fastpass traversing strategy.

Table 3 presents results for a nested decomposition solution of the previous model, where time differences are appreciated when infeasible subproblems appear. It has been solved aggregating periods to form two-period subproblems. This situation has been solved with the fastpass MIP strategy, the feasibility strategy and a combination that performs a fastpass MIP iteration after the linear solution and retakes feasibility strategy at the next iteration.

	Bd It	Sol. time	Exec. time	Quality
Fastpass MIP strategy	19	18484	2624	7e-4
Feasibility strategy	17	2705	566	14e-4
Combined strategy	21	5578	978	14e-4

Table 3. Comparison among strategies.

Finally, we present numerical results about stochastic problems solved with the goal of reaching a feasible solution. It is a stochastic version of the deterministic test problem we are dealing with, where stochasticity is introduced via a scenario tree. It is a four-scenario problem where the tree has been built up branching at periods 2 and 4. It is solved considering different periods as root nodes for the subtrees formed. These root nodes are reported on the first column of table 4. Remaining columns report iteration and time solutions as done at previous tables. Let us observe that the first row indicates that the whole problem cannot be solved without decomposition.

Decomposition	Feasibility strategy			
	Bd It	Sol time	Exec time	Quality
1	Out of memory			
1 2	2	408	294	3.2e-6
1 2 4	5	3258	2558	3.2e-6
1 2 4 6	17	2762	1056	158.1e-6
1 3 5	11	2554	1801	158.1e-6
1 2 3	5	648	340	80.6e-6

Table 4. Comparison among subtrees.

10 CONCLUSIONS

Lagrangean relaxation is a natural way to solve hydrothermal coordination problems within a Benders decomposition framework to approximate the recourse function. This paper has presented a nested Benders decomposition algorithm applied to stochastic problems that incorporate lagrangean relaxation to obtain correct dual variables for MIP problems.

High time solutions are due to time requested by the algebraic modeling language to communicate with a commercial solver. Time solution increases with the number of subproblems the problem has been divided into. It is thus recommended to create big subproblems instead of smaller ones in order to speed up convergence towards the solution.

Implemented lagrangean relaxation does not take advantage of repetitive solutions of the MIP problem. Considering lagrangean relaxation as a parametric MIP problem and solving it independently from an algebraic modeling language could reduce time solutions.

Tested traversing strategies suggest solving the problem with a linear strategy before introducing integrality conditions to the problem. They also recommend the use of a feasibility strategy for really big problems like those derived from stochastic situations.

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