Direct Computation of Derivatives in Production Cost Models based on Probabilistic Simulation

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Abstract

In this paper an efficient method is presented for calculating the derivatives of the production cost of a system with respect to the capacity of a unit, for the well-known probabilistic simulation method. Partial outages of generating units are taken into account, multiple blocks for each generating unit and dependencies involved among blocks are considered. The expression of the derivative of the production cost with respect to the commitment decision of a thermal unit is provided. A verification has been obtained by comparing the derivatives proposed in this paper with their numerical estimation for a simple case study. An application to the Scaled Down EPRI System D is also shown for comparison purposes.

Keywords

Marginal Costs, Derivatives, Sensitivities, Commitment Decision, Probabilistic Simulation, Production Cost, Operations Planning.

1 Introduction

The main objectives of production cost models are to evaluate the future system operation, to schedule the generation at minimum cost, and to coordinate the efficient use of limited resources. These models must properly represent the relevant decision variables and the actual operation of the power system.

Probabilistic simulation models, based on the well-known Booth-Baleriaux methodology [1, 4], are the most extensively used.

The derivative of the production cost determines the modification in variable operation cost due to a marginal change in a system parameter. The derivative with respect to the capacity of a generating unit is useful for generation expansion planning [3, 7, 12], maintenance scheduling [14], emissions dispatch [9], resources allocation, operations planning, and also in the context of competitive regulatory frameworks based on marginal prices. In this framework not only it is interesting to have the change in total variable cost with respect to the capacity of a unit but also with respect to its commitment decision.

The calculation of derivatives of the production cost with respect to unit parameters has been investigated in several papers. Bloom [2, 3] developed a recursive formula to compute the derivatives and used them for generation expansion planning solved by the generalized Bender's decomposition. Caramanis et al. [5] proposed an approach based on the cumulants method for a probabilistic simulation model. Fancher et al. [8] and Chen et al. [6] have improved this approach. Ramos et al. [7, 12] proposed a direct procedure to compute the derivatives in a deterministic production cost model used for static generation expansion planning. A refinement of this procedure and extension to a probabilistic simulation model is presented in this paper, based on research previously developed in [13]. More recently, Huang et al. [9] have also proposed a method based on direct computation of the derivatives. However, this paper extends their approach with the computation of the derivative of the production cost with respect to the commitment decision of a unit.

These derivatives have been used in a probabilistic simulation model specifically devoted to the optimization of the units loading order. This model resorts to branch and bound for solving the mixed integer programming problem [13]. The model has been applied to the Spanish electric power system. However, these derivatives can be useful in many other contexts as mentioned previously.

The method we present in this paper provides the derivatives in a compact way that can easily be understood and implemented. The presented approach includes partial outages of generating units, multiple blocks for each generating unit and dependencies among them, as well as a special treatment of the minimum load (technical minimum) of each unit. The loading order scheme dispatches the technical minima of the units first and the remaining blocks later. The expression of the derivative of the production cost with respect to the commitment decision of a thermal unit is provided.

It should be pointed out that the direct method of calculation of the derivatives proposed here can be integrated in a production cost model with any numerical or analytical approximation of the load-duration curve and convolution method used in a probabilistic simulation model: classical numerical methods, cumulant based methods or other analytical approximations (mix of normals, large deviation, equivalent energy function, etc.).

The paper is organized as follows. Firstly, the notation used along the paper is presented. Then the computation of the derivative of the production cost with respect to the commitment decision of a unit in the case of each unit loaded with a single block is given. Subsequently, the extension of this derivative for units with two loading blocks is shown. A verification is provided by comparing the derivatives proposed in this paper with their numerical estimation for a simple case study. Finally, an application to the Scaled Down EPRI

System D is shown for comparison purposes.

2 Notation

A_i commitment decision of unit i, the i-th in the loading order {0,1}.

E_i expected generation of unit i [MWh].

E_{K+1} expected non served energy [MWh].

E_{ij} expected produced energy by block j of the unit i [MWh].

$G_1(x)$ initial load-duration curve, customer demand.

$G_i(x)$ equivalent load-duration curve facing the unit i.

$G_{ij}(x)$ equivalent load-duration curve facing the block j of the unit i.
$G_{i1}(x) = G_i(x)$ and $G_{i3}(x) = G_{i+1}(x)$ assuming two blocks by unit.

$G^i_k(x)$ equivalent load-duration curve facing the unit k and then unit i is deconvolved.

$G^{i1}_{k1}(x)$ equivalent load-duration curve facing the must-run block of unit k and then the must-run block of unit i is deconvolved.

P_i capacity of unit i [MW].

P_{ij} capacity of block j of unit i [MW].
$\sum_{j=1}^{J} P_{ij} = P_i$

p_i availability of unit i [p.u.].

q_i equivalent forced outage rate of unit i [p.u.].
$p_i + q_i = 1$

q_{ij} partial forced outage rate of unit i from block j, including this [p.u.].
$p_i + \sum_{j=1}^{J} q_{ij} = 1$

T time period [h].

u_{i-1} loading point of unit i ($u_0 = 0$) [MW].

u'_k loading point of unit k after an increment in capacity of unit i [MW].

u^{j-1}_{i-1} loading point of the block j of the unit i ($u^0_{i-1} = u_{i-1}$) [MW].

v_i variable cost of unit i [\$/MWh].

v_{ij} variable cost of block j of unit i [\$/MWh].

v_{K+1} variable cost of non served energy [\$/MWh].

3 Commitment Decision Derivative with Single Block Units

This section presents the expression for the computation of the derivative of the production cost with respect to the commitment decision of a unit. In this section it is assumed that each unit can only be in either up or down state and is loaded as a single block. An extension of this derivative for units with two loading blocks is presented in the next section. Also, there are no hydro units in the power system.

Firstly, let us define the expected generation of a unit i in a probabilistic simulation model.

$$E_i = Tp_i \int_{u_{i-1}}^{u_i} G_i(x)dx \tag{1}$$

$$u_i = u_{i-1} + P_i = \sum_{k=1}^{i-1} P_k \tag{2}$$

The total production cost PC will be obtained from the expected generation of the units and the expected non served energy as follows.

$$PC = \sum_{k=1}^{K} v_k E_k + v_{K+1} E_{K+1} \tag{3}$$

Then any derivative with respect to the variable cost depends on the same derivative with respect to the expected produced energy by the units. If only one block is representing each unit the expected produced energy by a unit depends on the units previously loaded. Therefore, only units k loaded after unit i would be affected by a change in the capacity of unit i.

Let us calculate now the derivative of the expected generation of unit k with respect to the capacity of unit i. The expected generation of unit k is given by:

$$E_k = Tp_k \int_{u_{k-1}}^{u_k} G_k(x)dx \tag{4}$$

The equivalent load-duration curve faced by the unit k, $G_k(x)$, can be obtained from the convolution of unit i by the invariance property of the convolution, although the unit i is not contiguous to k in the loading order.

$$G_k(x) = p_i G^i_k(x) + q_i G^i_k(x - P_i) \tag{5}$$

We can take now an increment in the capacity of unit i and express again the expected generation of unit k

$$E'_k = Tp_k \int_{u'_{k-1}}^{u'_k} G'_k(x)dx \tag{6}$$

where

$$G'_k(x) = p_i G^i_k(x) + q_i G^i_k(x - P'_i) \tag{7}$$

$$P'_i = P_i + \Delta P_i \tag{8}$$

$$u'_k = P'_i + \sum_{j=1\ j\neq i}^{k-1} P_j \tag{9}$$

From equation (6) we can observe that the incremental impact on capacity of unit i affects:

- the loading and discharging points of the unit k, u'_{k-1} and u'_k, and
- the equivalent load-duration curve faced by the unit k, $G'_k(x)$.

After mathematical manipulation, detailed in Appendix A, we obtain the exact closed form for the derivative of the expected generation of unit k (loaded after unit i) with respect to the capacity of unit i.

$$\frac{\partial E_k}{\partial P_i} = T p_i p_k \left[G_k^i(u_k) - G_k^i(u_{k-1}) \right] \tag{10}$$

The derivative of the expected generation of unit k with respect to the commitment decision of unit i is obtained by multiplying by the capacity of unit i.

$$\frac{\partial E_k}{\partial A_i} = T p_i P_i p_k \left[G_k^i(u_k) - G_k^i(u_{k-1}) \right] \tag{11}$$

The marginal change in the commitment decision of unit i affects the loading points of unit k over the equivalent load-duration curve once the effect of failure of unit i has been discounted. Notice that the computation of $G_k^i(x)$ requires a deconvolution of unit i to be performed when the unit k is to be dispatched, with the consequent computational burden. Also it should be noted that the derivative is implicitly negative, that is, an increment in the commitment decision of a unit represents a decrement in the expected energy of units loaded later.

If unit k is dispatched earlier than unit i this derivative is zero.

The derivative of unit i with respect to its own commitment decision can be easily proved to be:

$$\frac{\partial E_i}{\partial A_i} = T p_i P_i G_i(u_i) \tag{12}$$

The effect in the expected non served energy can be thought as the summation of opposite effects with respect to all the units. Because of the invariance property of the convolution all the marginal changes in energy in all the units will be absorbed by the expected non served energy.

$$\frac{\partial E_{K+1}}{\partial A_i} = - \sum_{k=1}^{K} \frac{\partial E_k}{\partial A_i} \tag{13}$$

Then the derivative of the production cost with respect to the commitment decision of unit i has the following expression.

$$\begin{aligned} \frac{\partial PC}{\partial A_i} &= T p_i P_i \left\{ G_i(u_i)(v_i - v_{K+1}) \right. \\ &+ \left. \sum_{k=i+1}^{K} p_k \left[G_k^i(u_k) - G_k^i(u_{k-1}) \right] (v_k - v_{K+1}) \right\} \end{aligned} \tag{14}$$

that represents the effect with respect to the own unit i plus the effect with respect to the units k loaded after unit i plus the effect on the expected non served energy.

4 Commitment Decision Derivative with Two Block Units

The dispatch of the generating units when they have two blocks (or several ones, in general) is more involved. On one hand, dependencies among blocks have to be considered. That is, the effect of the failure of blocks of the same unit that were previously loaded must be eliminated from the equivalent load-duration curve, and therefore a deconvolution is required before the computation of the expected energy for the second block. On the other hand, the first loading block of each unit is a must-run block and has to be dispatched under the minimum load. Then all the technical minima of the units are loaded first, ordered by economic criterion, and immediately all the second blocks, also ordered by economic criterion. Schematically, this process is represented for two units, each one with two blocks in figure 1.

Figure 1: Loading process, expected energy computation and convolution.

The loading process of the units is as follows:

1. load the first must-run block (all the must-run blocks are ordered by increasing variable cost)
2. compute its expected generation

$$E_{i1} = T(1 - q_{i1}) \int_{u_{i-1}^0}^{u_{i-1}^1} G_{i1}(x) dx \tag{15}$$

$$u_{i-1}^1 = u_{i-1}^0 + P_{i1} \tag{16}$$

3. convolve the failure of the must-run block previously dispatched

$$G_{i2}(x) = (1 - q_{i1}) G_{i1}(x) + q_{i1} G_{i1}(x - P_{i1}) \tag{17}$$

4. repeat 1 to 3 for all the must-run blocks
5. load the second block with lower variable cost (all the second blocks are ordered by increasing variable cost)
6. deconvolve the must-run block corresponding to the previous second block

$$G_{i1}(x) = \frac{G_{i2}(x) - q_{i1} G_{i1}(x - P_{i1})}{1 - q_{i1}} \tag{18}$$

7. compute its expected produced energy

$$E_{i2} = T(1 - q_{i1} - q_{i2}) \int_{u_{i-1}^1}^{u_{i-1}^2} G_{i2}(x) dx \tag{19}$$

$$u_{i-1}^2 = u_{i-1}^1 + P_{i2} \tag{20}$$

8. convolve the failure of the must-run and second blocks simultaneously

$$
\begin{aligned}
G_{i3}(x) &= (1 - q_{i1} - q_{i2})G_{i1}(x) \\
&+ q_{i1}G_{i1}(x - P_i) + q_{i2}G_{i2}(x - P_{i2})
\end{aligned}
\tag{21}
$$

9. repeat 5 to 7 for all the second blocks

When the units have two blocks a change in the commitment decision of a unit affects both must-run and second blocks proportionally to the ratio between the capacity of each block and the total capacity of the unit. Therefore, the derivative of the production cost with respect to the commitment decision of the unit i has a much more complicated expression.

$$
\begin{aligned}
\frac{\partial PC}{\partial A_i} &= TP_{i1}\left\{(1-q_{i1})G_{i1}(u^1_{i-1})(v_{i1}-v_{K+1})\right. \\
&+ \sum_{k=i+1}^{K}(1-q_{k1})(1-q_{i1})[G^{i1}_{k1}(u^0_{k-1})-G^{i1}_{k1}(u^1_{k-1})](v_{k1}-v_{K+1}) \\
&+ \sum_{k=1}^{i-1}(1-q_{k1}-q_{k2})(1-q_{i1})[G^{i1}_{k2}(u^1_{k-1})-G^{i1}_{k2}(u^2_{k-1})](v_{k2}-v_{K+1}) \\
&- \left.(1-q_{i1}-q_{i2})G_{i2}(u^1_{i-1})(v_{i2}-v_{K+1})\right\} \\
&+ TP_i\left\{(1-q_{i1}-q_{i2})G_{i2}(u^2_{i-1})(v_{i2}-v_{K+1})\right. \\
&+ \sum_{k=i+1}^{K}(1-q_{k1}-q_{k2})(1-q_{i1}-q_{i2})[G^{i}_{k2}(u^1_{k-1})-G^{i}_{k2}(u^2_{k-1})](v_{k2}-v_{K+1}) \\
&+ \left.\sum_{k=i+1}^{K}(1-q_{k1}-q_{k2})q_{i2}\frac{P_{i1}}{P_i}[G^{i}_{k2}(u^1_{k-1}-P_{i2})-G^{i}_{k2}(u^2_{k-1}-P_{i2})](v_{k2}-v_{K+1})\right\}
\end{aligned} \tag{22}
$$

	1st Bl Capac [MW]	2nd Bl Capac [MW]	1st & 2nd Bls Var Cost [$/MWh]	1st Bl FOR	2nd Bl FOR
1	3	3	10	0.2	0.3
2	2	6	20	0.1	0.1
3	1	3	30	0.2	0.1
4	1	1	40	0.2	0.2

Table 1: Characteristics of the test system.

The previous expression is composed of the following terms:

- the first term considers the marginal changes in the production cost of the must-run block of unit i plus changes in the must-run blocks of units k loaded after must-run block of unit i plus changes in the second blocks of units k loaded before second block of unit i plus the change in the second block of unit i.
- and the second term takes into account the marginal change in second block of unit i plus changes in second blocks of units k loaded after second block of unit i.

As before, the marginal change in expected non served energy affects all the terms.

5 Verification

The calculation of the derivatives has been implemented in a production cost simulation model using MATLAB [10]. This model uses both a numerical and analytical (cumulants method) approximation for the convolution of the load-duration curve.

A very simple case study has been used to test the new computation method. Table 5 describes the characteristics of the system units. The system load is assumed to have a trapezoidal distribution with a minimum load of 8 MW and a maximum load of 16 MW. The load-duration curve is approximated in the numerical method by 16000 points. The time period is 1000 h. The expected non served energy is penalized at a variable cost of 100 $/MWh.

Deliberately, in this case study it has been assumed a high forced outage rate for each unit in order to clearly show the differences between the theoretical computation and the numerical approximation of the derivatives.

The calculation of the derivatives was done using the formulas developed and presented in section 4 and also by numerical perturbation when an increment was added to the commitment decision of each particular unit. Table 5 presents the results, which happen to be very similar. The relative errors are always below 0.05 %. The increment given to the perturbation corresponds to one point in the definition of the load-duration curve, then no approximation is used for the computation of the energy generated by the units and only numerical errors due to the convolution process could be possible.

To analyze in more detail each term involved in the mathematical expression of the derivative, it is shown the effect of the derivative of the commitment decision of unit 2 on each loading block of each unit in table 5.

	Numerical Approximation	Theoretical Computation
1	-107728	-107745.75
2	-95966	-96016
3	-46305	-46327.25
4	-13735	-13740

Table 2: Derivative with respect to the commitment decision of each unit.

	Numerical Approximation	Theoretical Computation
1st Bl Unit 1	0	0
1st Bl Unit 2	9000	9000
1st Bl Unit 3	0	0
1st Bl Unit 4	0	0
2nd Bl Unit 1	-900.28	-900
2nd Bl Unit 2	2445.5	2450
2nd Bl Unit 3	-4275.4	-4277
2nd Bl Unit 4	-925.1	-925.5
EENS	-5344.7	-5372.7

Table 3: Derivative of the expected generation with respect to the commitment decision of unit 2.

6 Application

In this section we use a synthetic utility system called Scaled Down EPRI System D developed in [11] and also used in testing the derivatives computation in [8] and [9].

The units of the system are presented in table 6. There are 52 units of eight types. The demand is normally distributed with mean 6267.4 MW and standard deviation 745.4 MW. The time period is 728 h. The loading order of the units is based on variable costs.

Table 6 shows some derivatives of the production cost with respect to the commitment decision of unit 8.

Loading Order	Unit Size [MW]	No. of Units	Variable Cost [$/MWh]	FOR [p.u.]
1	1200	2	10	0.15
2	800	1	15	0.24
3	600	2	18	0.21
4	400	2	20	0.13
5	400	1	50	0.13
6	200	8	55	0.074
7	200	7	60	0.074
8	50	29	70	0.24

Table 4: Scaled Down EPRI System D.

	Theoretical Computation
7	-9098690
8	-1707435
9	-275758
17	74712

Table 5: Derivative with respect to the commitment decision of unit 8.

7 Conclusions

In this paper a method for calculating the derivatives of the production cost of a system with respect to the capacity of a unit for the well-known probabilistic simulation method has been presented. Partial outages of generating units are taken into account, multiple blocks for each generating unit and dependencies involved among blocks have been considered. The expression of the derivative of the production cost with respect to the commitment decision of a thermal unit has been provided. These derivatives can be used for generation expansion planning, maintenance scheduling, operations planning or in the context of competitive regulatory frameworks based on marginal prices. A verification is provided by comparing the derivatives proposed in this paper with their numerical estimation for a simple case study. An application to the Scaled Down EPRI System D has been also shown for comparison purposes.

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Appendix A

In this Appendix it follows the proof of the expression for the derivative of the expected generation of a unit k with respect to the capacity of unit i, previously loaded, when each unit is modeled as a single block.

$$\begin{aligned} E'_k &= Tp_k \int_{u'_{k-1}}^{u'_k} G'_k(x)dx \\ &= -Tp_k \int_{u_{k-1}}^{u'_{k-1}} G'_k(x)dx + Tp_k \int_{u_{k-1}}^{u_k} G'_k(x)dx \\ &\quad + Tp_k \int_{u_k}^{u'_k} G'_k(x)dx \end{aligned} \tag{23}$$

$$E_k = Tp_k \int_{u_{k-1}}^{u_k} G_k(x)dx \tag{24}$$

$$\frac{\partial E_k}{\partial P_i} = \frac{E'_k - E_k}{\Delta P_i} \tag{25}$$

$$\int_{u_{k-1}}^{u_k} G'_k(x)dx =$$
$$\int_{u_{k-1}}^{u_k} [p_i G^i_k(x) + q_i G^i_k(x - P'_i)]dx =$$
$$p_i \int_{u_{k-1}}^{u_k} G^i_k(x)dx + q_i \int_{u_{k-1}}^{u_k} G^i_k(x - P'_i)dx =$$
$$p_i \int_{u_{k-1}}^{u_k} G^i_k(x)dx + q_i \int_{u_{k-1}-\Delta P_i}^{u_k-\Delta P_i} G^i_k(x - P_i)dx =$$
$$p_i \int_{u_{k-1}}^{u_k} G^i_k(x)dx + q_i \int_{u_{k-1}}^{u_k} G^i_k(x - P_i)dx +$$
$$q_i \int_{u_{k-1}-\Delta P_i}^{u_{k-1}} G^i_k(x - P_i)dx - q_i \int_{u_k-\Delta P_i}^{u_k} G^i_k(x - P_i)dx$$

$$\begin{aligned} \int_{u_{k-1}}^{u_k} G'_k(x)dx &= \int_{u_{k-1}}^{u_k} G_k(x)dx \\ &+ q_i \int_{u_{k-1}-\Delta P_i}^{u_{k-1}} G^i_k(x - P_i)dx \\ &- q_i \int_{u_k-\Delta P_i}^{u_k} G^i_k(x - P_i)dx \end{aligned} \tag{26}$$

If we apply the previous result to the other terms of equation (23) we obtain:

$$\begin{aligned} \int_{u_{k-1}}^{u'_{k-1}} G'_k(x)dx &= \int_{u_{k-1}}^{u'_{k-1}} G_k(x)dx \\ &+ q_i \int_{u_{k-1}-\Delta P_i}^{u_{k-1}} G^i_k(x - P_i)dx \\ &- q_i \int_{u_{k-1}}^{u'_{k-1}} G^i_k(x - P_i)dx \end{aligned}$$

$$\begin{aligned} \int_{u_k}^{u'_k} G'_k(x)dx &= \int_{u_k}^{u'_k} G_k(x)dx \\ &+ q_i \int_{u_k-\Delta P_i}^{u_k} G^i_k(x - P_i)dx \\ &- q_i \int_{u_k}^{u'_k} G^i_k(x - P_i)dx \end{aligned} \tag{27}$$

then

$$\begin{aligned} E'_k &= Tp_k \Bigg\{ -\int_{u_{k-1}}^{u'_{k-1}} G_k(x)dx \\ &+ \int_{u_{k-1}}^{u_k} G_k(x)dx \\ &+ \int_{u_k}^{u'_k} G_k(x)dx \\ &+ q_i \int_{u_{k-1}}^{u'_{k-1}} G^i_k(x - P_i)dx \\ &- q_i \int_{u_k}^{u'_k} G^i_k(x - P_i)dx \Bigg\} \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{\partial E_k}{\partial P_i} &= Tp_k \{G_k(u_k) - G_k(u_{k-1}) \\ &+ q_i \left[G^i_k(u_{k-1} - P_i) - G^i_k(u_k - P_i)\right]\Big\} \end{aligned} \tag{29}$$

Reordering the equation (5) we can obtain

$$p_i G^i_k(x) = G_k(x) - q_i G^i_k(x - P_i) \tag{30}$$

and introducing this result in (29) we obtain the following expression

$$\frac{\partial E_k}{\partial P_i} = Tp_i p_k \left[G^i_k(u_k) - G^i_k(u_{k-1})\right] \tag{31}$$

This result is similar to the one in [9].