NONLINEAR OPTIMIZATION PROBLEM SET

PROBLEM. INVENTORIES

A storekeeper manages two items A1 and A2 whose demands are produced regularly and constantly. Its total annual values are respectively N_1 and N_2 and its total annual storage costs are respectively $cs1$ and $cs2$.

An agreement with the supplier of those items provides that:

- The replenishment orders will be simultaneous.
- Each global order of both items will have a cost order of cl.
- The global order will not exceed 4000 units and will be made up of at least 1200 units of A1 and 1000 units of A2.

a) Develop an optimization model that minimizes the managing cost and propose the necessary conditions for an optimal solution.

b) Is there a global minimum? What type of point is $(1200, 2500)$ for $N_2 = 50000$, $N_1 = 24000$, $cl = 500\,€$, $cs_1 = 50\ €/_{unidad\ año}$ and $cs_2 = 150\ €/_{unidad\ año}$?

PROBLEM. CUBIC FUNCTION WITH HYPERPLANES

Use KKT conditions to obtain an optimal solution of problem

$$\begin{aligned}
&\min f(x) = x_1^{\ 3} - 2x_1 - x_2 \\
&x_1 + x_2 \le 1 \\
&x_1 \ge 0 \\
&x_2 \ge 0
\end{aligned}$$

PROBLEM. CHECK POINT

Calculate

$$\begin{aligned}
&\min z = y^2 + x \\
&xy \ge 2 \\
&x^2 + 2y^2 \le 6 \\
&x \ge 0
\end{aligned}$$

a) Establish the Karush-Kuhn-Tucker condition of the problem

b) Check the following points, $A = (1.6, 1.3)$, $B = \left(\sqrt{2}, \sqrt{2}\right)$ and $C = (2,1)$ as optimum candidates, indicating if applicable what kind of optimum is reached.

PROBLEM. LINEAR OBJECTIVE

Maximize the following problem with nonlinear constraints

$$\max f(x) = -20x_1 - 10x_2$$
$$x_1^2 + x_2^2 \le 1$$
$$x_1 + 2x_2 \le 2$$
$$x_1 \ge 0, x_2 \ge 0$$

a) Establish the Lagrangian
b) Establish the necessary and sufficient conditions of the problem
c) Characterize the following points (1,0), (0,1) and $\left(\frac{10}{\sqrt{125}}, \frac{5}{\sqrt{125}}\right)$ for this optimization problem?

ADDITIONAL PROBLEMS

PROBLEM. PROBLEM 2

A power company faces a lawsuit divided into intervals of peak and valley load. If a price of p_1 €/MWh is charge at the time of peak load, customers will asked for $60 - 0.5p_1$ MWh of energy. If a price of p_2 €/MWh is charge at the time of valley load, customers will asked for $40 - p_2$ MWh of energy. The company must have enough capacity to satisfy the demand during both intervals. The cost of production of the company is 100 €/MWh. Determine how the company can optimize its contribution margin (revenue-costs).

PROBLEM. PROBLEM 3

A firm produces and sells two products A and B at prices p_A and p_B and quantities Q_A and Q_B. The quantities sold are connected to prices by the following demand functions $Q_A = 60 - 21p_A + 0.1p_B$ and $Q_B = 50 - 25p_B + 0.1p_A$.

Production costs per unit of product are 0.2 and 0.3 and the available production capacity of each of them is 25 and 50, respectively. Furthermore, there is a limit on the total capacity of production of the firm represented by the inequality: $Q_A + 2Q_B \le 50$.

Formulate and solve the problem of determining the values that maximize the contribution margin (gross profit) of the firm.

PROBLEM. PROBLEM 4 PART A

$$
\begin{aligned}
&\min (x-9/4)^2+(y-2)^2 \\
&x^2-y \le 0 \\
&x+y-6 \le 0 \\
&-x \le 0 \\
&-y \le 0
\end{aligned}
$$

PROBLEM. PROBLEM 4 PART B

Demonstrate that points $A=(1,2)$, $B=(2,-2)$ and $C=(0,0)$ are global maxima of the following nonlinear optimization problem.

$$
\begin{aligned}
&\max \left(x-13/6\right)^2+\left(y-1/6\right)^2 \\
&4x^2+3y-10 \le 0 \\
&-2x+y \le 0 \\
&x+y \ge 0
\end{aligned}
$$

PROBLEM. PROBLEM 4 PART C

$$
\begin{aligned}
&\min -3x+y-z^2 \\
&x+y+z \le 0 \\
&-x+2y+z^2=0
\end{aligned}
$$

PROBLEM. TRIATHLON

A participant in a triathlon (swimming races trails, biking and cross) is provided with a waterproof bag with 1600 grams of supplies that should manage to obtain good times in races.

In the swimming race, the relationship between ingested food x_1 and time to finish it is

$$t_{natacion}=\frac{4000}{x_1}, 100 \le x_1 \le 1000$$

In the bike race, the relation between ingested food x_2 and time of the race is:

$$t_{ciclista}=90-\left(\frac{x_2}{50}\right)^2, 100 \le x_2 \le 400$$

while in the cross race that relation is

$$t_{cross} = \sqrt{1600 - x_3}, 400 \le x_3 \le x_1$$

Check if

$$x_1 = 700, x_2 = 400, x_3 = 500$$

can be an optimal solution of this problem.

PROBLEM. CUBIC FUNCTION

$$\min x_1^2 - 6x_1 + x_2^3 - 3x_2$$
$$x_1 + x_2 \le 1$$
$$x_1 \ge 0, x_2 \ge 0$$

It is asked to:

a) Establish the sufficient KKT conditions and, if necessary, the sufficient and necessary conditions

b) Calculate a minimum, if possible, by the fulfillment of the KKT conditions

c) Indicate what type of point is $\left(\frac{1}{2}, \frac{1}{2}\right)$

PROBLEM. SCIENCE OF COMPLEXITY

A student of Science of Complexity has to prepare two exams about subjects Theory of Fractals (TF) and Theory of Self-organization (TA). He has 20 hours for preparing both exams.

Dedicating x hours studying TF is a score of $2.5\sqrt{x}$. In case of studding TA de score is $\frac{5}{1.5}\log_4 x \; (x \ge 1)$.

The primary objective of the student is to pass (obtain 5 points) both subjects, and when this objective is guaranteed, to obtain a global punctuation as higher as possible. The highest punctuation in a subject is 10 points.

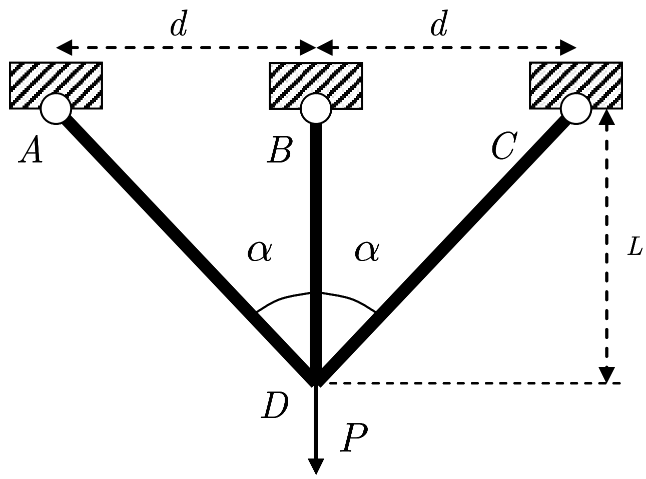
a) Formulate the optimization model and the associated Karush-Kuhn-Tucker conditions

b) Prove that the optimal solution distributes 12 hours to TF and 8 to TA

Note: Remember that if $y = \log_a x \Rightarrow y' = \frac{1}{\ln a}\frac{1}{x}$

PROBLEM. SIMPLE STRUCTURE

We have a simple structure that consists of three rods AD, BD and CD which are attached to the roof in points A, B and C and bonded together in point D. Points A, B and C are fixed to a previous structure that sets the maximum distance between them to a value d of 3 meters.



We want to determine what should be the angle α and section A equal for all rods and length L of central rod to minimize the total cost of the structure. We know that price p per m^3 of the used steel is 7300 €, its allowable stress σ_{adm} is 235 10^6 N/m^2 and force P (the one that the structure has to support) is 30 kN.

Nonlinear optimization: Class problems solutions

Solution. Inventories

a) Develop an optimization model that minimizes the managing cost and propose the necessary conditions for an optimal solution.

If we denote n_1 and n_2 as the requested amounts of each item, the managing cost function has two components: an order cost equal to the number of orders r

$$r = \frac{N_1}{n_1} = \frac{N_2}{n_2}$$

times the cost of making an order cl (as the orders are made simultaneously the same amount of orders will be made for each item) and a storage cost that will be equal to:

$$\frac{1}{2}\left(n_1 cs_1 + n_2 cs_2\right)$$

Therefore, the cost function will be:

$$G\left(n_1, n_2\right) = \frac{N_1}{n_1} cl + \frac{1}{2}\left(n_1 cs_1 + n_2 cs_2\right)$$

So the following optimization problem has to be solved:

$$\begin{aligned} & \min G\left(n_1, n_2\right) \\ & n_1 N_2 = n_2 N_1 \\ & n_1 + n_2 \leq 4000 \\ & n_1 \geq 1200 \\ & n_2 \geq 1000 \end{aligned}$$

having associated the Lagrange function

$$\begin{aligned} L\left(n_1, n_2; \mu, \lambda_1, \lambda_2, \lambda_3\right) = {} & \frac{N_1}{n_1} cl + \frac{1}{2}\left(n_1 cs_1 + n_2 cs_2\right) + \mu\left(n_1 N_2 - n_2 N_1\right) + \\ & + \lambda_1\left(n_1 + n_2 - 4000\right) + \lambda_2\left(1200 - n_1\right) + \lambda_3\left(1000 - n_2\right) \end{aligned}$$

The necessary conditions for Karush-Kuhn-Tucker optimal are:

$$-\frac{N_1}{n_1^{\,2}}cl+\frac{1}{2}cs_1+\mu N_2+\lambda_1-\lambda_2=0$$
$$\frac{1}{2}cs_2-\mu N_1+\lambda_1-\lambda_3=0$$
$$\lambda_1\left(n_1+n_2-4000\right)=0$$
$$\lambda_2\left(1200-n_1\right)=0$$
$$\lambda_3\left(1000-n_2\right)=0$$
$$n_1N_2=n_2N_1$$
$$n_1+n_2\leq 4000$$
$$1200-n_1\leq 0$$
$$1000-n_2\leq 0$$
$$\lambda_1,\lambda_2,\lambda_3\geq 0$$

b) Is there a global minimum? What type of point is $(1200,2500)$ for $N_2 = 50000$, $N_1 = 24000$, $cl=500\,€$, $cs_1=50\ €/_{unidad\ año}$ and $cs_2=150\ €/_{unidad\ año}$?

The objective function has a positive semidefinite Hessian in the feasible region of the problem. The problem constraints establish a convex feasible region since all the inequality constraints are linear. Therefore, you can apply the necessary and sufficient conditions that establish that there is a single global optimum of the problem. Besides, by the Weirstrass theorem we can say that it is a global optimum because the o.f. is bounded and the feasible region is also bounded.

It is verified that point $(1200,2500)$ is a global optimum of the problem with a cost of inventory management of 227500 €. The Lagrange multipliers are: μ=0.003125, λ_1=0, λ_2=172.92, λ_3 =0.

SOLUTION. CUBIC FUNCTION WITH HYPERPLANES.

Expressing the problem in its standard formulation inequality constraints must be all lower or equal. Therefore:

$$\min f(x) = x_1^{\ 3} - 2x_1 - x_2$$
$$x_1 + x_2 \le 1$$
$$-x_1 \le 0$$
$$-x_2 \le 0$$

The lagrangian has the following expression:

$$L(x) = x_1^3 - 2x_1 - x_2 + \lambda_1 \left(x_1 + x_2 - 1\right) - \lambda_2 x_1 - \lambda_3 x_2$$

The KKT conditions with inequality constraints are:

$$3x_1^2 - 2 + \lambda_1 - \lambda_2 = 0$$
$$-1 + \lambda_1 - \lambda_3 = 0$$
$$\lambda_1 \left(x_1 + x_2 - 1\right) = 0$$
$$\lambda_2 x_1 = 0$$
$$\lambda_3 x_2 = 0$$
$$x_1 + x_2 \le 1$$
$$-x_1 \le 0$$
$$-x_2 \le 0$$

Solving the hypothesis that $\lambda_3 = 0$ then from the second equation we obtain $\lambda_1 = 1$ and therefore $x_1 + x_2 = 1$. Making de assumption that $\lambda_2 = 0$ and taking into account that $\lambda_1 = 1$ we obtain $x_1 = \dfrac{1}{\sqrt{3}}$ and therefore $x_2 = 1 - \dfrac{1}{\sqrt{3}}$. This is a point that satisfies the necessary conditions and as the feasible region is convex and differentiable like the objective function in that region, the obtained point applying the necessary and sufficient conditions is a global minimum.

Solution. Check points

a) Establish the Karush-Kuhn-Tucker condition of the problem

The problem in its standard format formulating the inequations with $\leq$ sense, reaching the following problem:

$$\begin{aligned} \min z &= y^2 + x \\ xy &\geq 2 \\ x^2 + 2y^2 &\leq 6 \\ x &\geq 0 \end{aligned}$$

The Lagrangian of the standard format problem has the following expression:

$$L(x, y, \lambda_1, \lambda_2, \lambda_3) = y^2 + x + \lambda_1(-xy + 2) + \lambda_2(x^2 + 2y^2 - 6) + \lambda_3(-x)$$

The necessary KKT conditions for a local minimum are indicated below. The objective function is convex, but the feasible region is not convex due to the first constraint.

$$\begin{aligned} &1)\ 1 - \lambda_1 y + 2\lambda_2 x - \lambda_3 = 0 \\ &2)\ 2y - \lambda_1 x + 4\lambda_2 y = 0 \\ &3)\ \lambda_1(-xy + 2) = 0 \\ &4)\ \lambda_2(x^2 + 2y^2 - 6) = 0 \\ &5)\ \lambda_3(-x) = 0 \\ &\textit{Punto factible} \\ &\lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}$$

b) Check the following points as possible candidates for the mentioned optimum indicating, if applicable, what kind of optimum is reached.

$$\begin{aligned} A &= (1.6, 1.3) \\ B &= (\sqrt{2}, \sqrt{2}) \\ C &= (2, 1) \end{aligned}$$

Point A

The compliance of conditions 3, 4 and 5 make λ_1, λ_2 and λ_3 to be zero. However, these values do not satisfied conditions 1 and 2 so point A is not an important point.

Point B

The compliance of condition 5 makes λ_3 to be zero. Conditions 3 and 4 allow λ_1 and λ_2 to be different from zero. The compliance of conditions 1 and 2 make λ_1 and λ_2 to be negative so point B is not important.

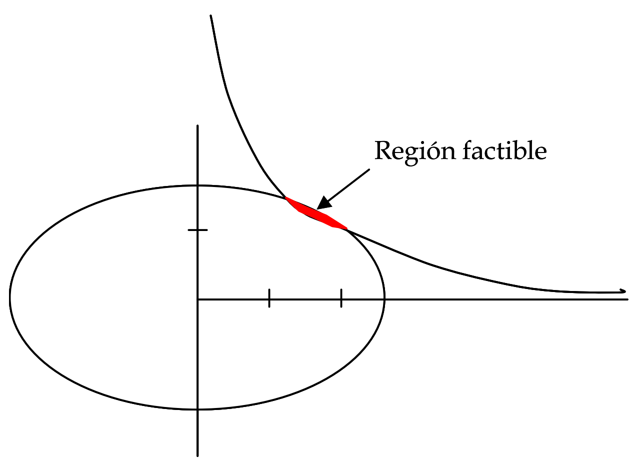
Point C

The compliance of conditions 3, 4 and 5 make λ_1 and λ_2 to be different from zero and make λ_3 to be zero. Conditions 1 and 2 allow to calculate the values of λ_1 and λ_2 solving the following system of equations:

$$\left.\begin{array}{l} 1-\lambda_1+4\lambda_2=0 \\ 2-2\lambda_1+4\lambda_2=0 \end{array}\right\} \lambda_1=1, \lambda_2=0$$

This point complies with the necessary KKT conditions and therefore it can be said that is a candidate for local minimum. However, since the first restriction is not convex as it is proved by the determinant of its Heassian, the KKT sufficient conditions are not met and therefore we cannot claim that the point is a global minimum.

However, beyond the resolution of this problem, it is found that point C is a global minimum. This is because there are no other minimum candidates in a feasible region, as it is shown in the following figure, and applying the **Weierstrass Theorem.** This theorem establishes that a continuous function in a feasible and closed region reaches its maximum and minimum values in points of the region. This concludes that point (2,1) is global minimum.



SOLUTION. LINEAR OBJECTIVE

a) Establish the Lagrangian

First the problem is formulated in its standard form:

$$\min f(x) = 20x_1 + 10x_2$$
$$x_1^2 + x_2^2 - 1 \leq 0$$
$$x_1 + 2x_2 - 2 \leq 0$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

The resulting lagrangian is:

$$L(x,\lambda) = 20\,x_1 + 10\,x_2 + \lambda_1\left(x_1^2 + x_2^2 - 1\right) + \lambda_2\left(x_1 + 2x_2 - 2\right) + \lambda_3\left(-x_1\right) + \lambda_4\left(-x_2\right)$$

b) Establish the necessary and sufficient conditions of the problem

As $f(x)$ is a plane, is a convex function and both constraints have a positive definite Hessian and therefore are convex. The necessary and sufficient conditions are the following:

$$\nabla f(x) + \sum_{i=1}^{4} \lambda_i \nabla g_i(x) = 0$$
$$\lambda_i \nabla g_i(x) = 0 \quad \forall i$$
$$\lambda_i \geq 0$$
$$\textit{Punto factible}$$

Any point that satisfies these necessary conditions will be global minimum. Numerically these conditions are written taking into account the value of the gradients of the objective function and the constraints:

$$\nabla f\left(x\right) = \begin{bmatrix} 20 \\ 10 \end{bmatrix} \quad \nabla g_1(x) = \begin{bmatrix} 2\,x_1 \\ 2\,x_2 \end{bmatrix} \nabla g_2(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \nabla g_3(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \nabla g_4(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$20+2\lambda_1 x_1+\lambda_2-\lambda_3=0$$
$$10+2\lambda_1 x_2+2\lambda_2-\lambda_4=0$$
$$\lambda_1\left(x_1^2+x_2^2-1\right)=0$$
$$\lambda_2\left(x_1+2x_2-2\right)=0$$
$$\lambda_3\left(-x_1\right)=0$$
$$\lambda_4\left(-x_2\right)=0$$
Punto factible
$$\lambda_i\geq 0; i=1,\ldots,4$$

c) What are in this problem points (1,0), (0,1) and $\left(\frac{10}{\sqrt{125}},\frac{5}{\sqrt{125}}\right)$?

c.1) Point (1,0)

From (4) $\lambda_2=0$
From (5) $\lambda_3=0$
From (1) $\lambda_1<0$

This problem is feasible but is not important as it is a negative multiplier

c.2) Point (0,1)

From (6) $\lambda_4=0$

From (1) and (2) we obtain the following equation system:

$$20+\lambda_2-\lambda_3=0$$
$$10+2\lambda_1+2\lambda_2=0$$

Assuming $\lambda_3=0$ then $\lambda_2=-20$
Assuming $\lambda_3\neq 0$ and $\lambda_1=0$ then $\lambda_2=-5$
Assuming $\lambda_3\neq 0$, $\lambda_1\neq 0$ and $\lambda_2=0$ then $\lambda_1=-5$

Therefore, this point is feasible but is not important as the values of the multipliers are negative.

c.3) Point $\left(\frac{10}{\sqrt{125}},\frac{5}{\sqrt{125}}\right)$

From (5) and (6) we obtain: $\lambda_3=0,\lambda_4=0$
From (4) we obtain: $\lambda_2=0$
From (1) and (2) we obtain: $\lambda_1=-\sqrt{125}$

This point is feasible but is not important as there is a negative multiplier

ADDITIONAL PROBLEMS SOLUTIONS

Solution Problem 2

$$
\begin{aligned}
&c\arg a\_punta \Rightarrow p_1 €/MW.h \to d_1 = 60 - 0'5p_1 \\
&c\arg a\_valle \Rightarrow p_2 €/MW.h \to d_2 = 40 - p_2 \\
&\cos te\_de\_produccion \Rightarrow 100€/MW.h
\end{aligned}
$$

$$
\begin{aligned}
&C\cos tes : [100(60 - 0.5p_1) + 100(40 - p_2)] \\
&Ingresos : [p_1(60 - 0.5p_1) + p_2(40 - p_2)]
\end{aligned}
$$

We want to maximize profit that is the difference between revenues and costs, so the objective function is:

$$
\begin{aligned}
&\max(ingresos - \cos tes) = \\
&\max([p_1(60 - 0.5p_1) + p_2(40 - p_2)] - [100(60 - 0.5p_1) + 100(40 - p_2)]) = \\
&\max(p_1 - 100)(60 - 0.5p_1) + (p_2 - 100)(40 - p_2)
\end{aligned}
$$

This objective function is limited with some constraints, meeting the demand both in peak hours and in valley hours:

$$
\left.\begin{aligned} 60 - 0.5p_1 \geq 0 \\ 40 - p_2 \geq 0 \end{aligned}\right\} \Rightarrow \begin{cases} p_1 \leq 120 \\ p_2 \leq 40 \end{cases}
$$

Operating the objective function the problem is reduced to:

$$
\begin{aligned}
&\max(60 - 0.5p_1)(p_1 - 100) + (40 - p_2)(p_2 - 100) = \\
&\max(60p_1 - 0.5p_1^2 - 6000 + 50p_1 + 40p_2 - p_2^2 - 4000 + 100p_2) = \\
&\max(110p_1 - 0.5p_1^2 + 140p_2 - p_2^2 - 10000)
\end{aligned}
$$

$$
\begin{aligned}
&\max z = (110p_1 - 0,5p_1^2 + 140p_2 - p_2^2 - 10000) \\
&p_1 \leq 120 \\
&p_2 \leq 40
\end{aligned}
$$

$$
\Rightarrow \min -z = 0,5p_1^2 + p_2^2 - 110p_1 - 140p_2 + 10000
$$

$$
s.a. \begin{cases} p_1 \leq 120 \\ p_2 \leq 40 \end{cases} \Bigg\| \begin{aligned} &\lambda_1 \to p_1 - 120 \leq 0 \\ &\lambda_2 \to p_2 - 40 \leq 0 \end{aligned}
$$

With KKT conditions:

$$\nabla z(x,y) + \sum \lambda_i \nabla g_i(x,y) = 0$$
$$\lambda_i g_i(x,y) = 0$$
$$\lambda_i \geq 0$$
$$\nabla z(p_1,p_2) = \begin{bmatrix} p_1 - 110 \\ 2p_2 - 140 \end{bmatrix}$$
$$\nabla g_1(p_1,p_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\nabla g_2(p_1,p_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\begin{cases} p_1 - 110 + \lambda_1 = 0 \\ 2p_2 - 140 + \lambda_2 = 0 \\ (p_1 - 120)\lambda_1 = 0 \\ (p_2 - 40)\lambda_2 = 0 \\ \lambda_1 ; \lambda_2 \geq 0 \end{cases}$$
$$Si : \lambda_1 \neq 0$$
$$p_1 = 120$$
$$\rightarrow \lambda_1 = -10 \Rightarrow ERROR$$

$$Si : \lambda_1 = 0$$
$$p_1 = 110$$
$$\rightarrow \begin{cases} Si : \lambda_2 \neq 0 \\ p_2 = 40 \Rightarrow \lambda_2 = 60 \end{cases}$$
$$\rightarrow \begin{cases} Si : \lambda_2 = 0 \\ p_2 = 70 \Rightarrow ERROR \rightarrow p_2 \leq 40 \end{cases}$$
$$SOLUCION :$$
$$\boxed{\begin{matrix} p_1 = 110 \\ p_2 = 40 \end{matrix}} \rightarrow \boxed{\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 60 \end{matrix}}$$

The maximum (global because of the KKT sufficient conditions) is reached at values $(p_1, p_2) = (110, 40)$ (multipliers $u_1 = 0$, $u_2 = 60$), profit = 50€.

SOLUTION. PROBLEM 3

We want to maximize the gross profit of the firm that produces two products A and B. The gross profit is the difference between net income and net costs. We have constraints on total production capacity and available production of each product. The problem is formulated as follows:

$$Costes \rightarrow C_A Q_A + C_B Q_B$$
$$Beneficios \rightarrow p_A Q_A + p_B Q_B$$

$$\max (p_A - C_A) Q_A + (p_B - C_B) Q_B$$

$$\min (C_A - p_A) Q_A + (C_B - p_B) Q_B$$
$$s.a. \left| \begin{array}{l} Q_A + 2Q_B \leq 50 \\ Q_A \leq 25 \\ Q_B \leq 50 \end{array} \right.$$

$$Sabiendo\,que:$$
$$Q_A = 60 - 21 p_A + 0.1 p_B$$
$$Q_B = 50 - 25 p_B + 0.1 p_A$$

$$Operamos:$$
$$\min (0.2 - p_A)(60 - 21 \cdot p_A + 0.1 p_B) + (0.3 - p_B)(50 - 25 \cdot p_B + 0.1 p_A)$$
$$\min (21 p_A{}^2 - 64.17 p_A - 0.2 p_A p_B + 25 p_B^2 - 57.48 p_B + 27)$$
$$s.a. \left| \begin{array}{l} 60 - 21 \cdot p_A + 0.1 p_B + 2 \cdot (50 - 25 p_B + 0.1 p_A) - 50 \leq 0 \\ 60 - 21 \cdot p_A + 0.1 p_B - 25 \leq 0 \\ 50 + 0.1 \cdot p_A - 25 p_B - 50 \leq 0 \end{array} \right.$$

$$\min(21{p_A}^2 - 64.17p_A - 0.2p_Ap_B + 25p_B^2 - 57.48p_B + 27)$$

$$\left|\begin{array}{ll} -20.8p_A - 49.9p_B + 110 \le 0 & \Rightarrow \lambda_1 \\ -21p_A + 0.1p_B + 35 \le 0 & \Rightarrow \lambda_2 \\ 0.1p_A - 25p_B + \le 0 & \Rightarrow \lambda_3 \end{array}\right.$$

$$\begin{aligned} &Con\, las\, condiciones\, KKT: \\ &\nabla z(p_A, p_B) + \sum \lambda_i \nabla g_i(p_A, p_B) = 0 \\ &\lambda_i g_i(p_A, p_B) = 0 \\ &\lambda_i \ge 0 \end{aligned}$$

$$\nabla z(p_A, p_B) = \begin{bmatrix} 42p_A - 64.17 - 0.2p_B \\ -0.2p_A + 50p_B - 57.48 \end{bmatrix}$$

$$\nabla g_1(p_A, p_B) = \begin{bmatrix} -20.8 \\ -49.9 \end{bmatrix} \quad \nabla g_2(p_A, p_B) = \begin{bmatrix} -21 \\ 0.1 \end{bmatrix} \quad \nabla g_3(p_A, p_B) = \begin{bmatrix} 0.1 \\ -25 \end{bmatrix}$$

The KKT conditions are turned into the following expressions:

$$\begin{aligned} &42p_A - 0.2p_B - 20.8\lambda_1 - 21\lambda_2 + 0.1\lambda_3 - 64.17 = 0 \\ &-0.2p_A + 50p_B - 49.9\lambda_1 + 0.1\lambda_2 - 25\lambda_3 - 57.48 = 0 \\ &\lambda_1(-20.8p_A - 49.9p_B + 110) = 0 \\ &\lambda_2(-21p_A + 0.1p_B + 35) = 0 \\ &\lambda_3(0.1p_A - 25p_B) = 0 \\ &\lambda_1, \lambda_2, \lambda_3 \ge 0 \end{aligned}$$

Solving the equation system assuming hypothetical values of the Lagrange multipliers we obtain a global minimum of the problem as all KKT necessary and sufficient conditions fulfilled.

We start assuming non-zero values in two of the three Lagrange multipliers.

$$\left.\begin{array}{l} \lambda_3 \ne 0 \rightarrow [5]: p_A = 250p_B \\ \lambda_2 \ne 0 \rightarrow [4]: -21p_A + 0.1p_B + 35 = 0 \end{array}\right\} p_A = 1.666; p_B = 0.00666$$

$$\left.\begin{array}{r} \lambda_1 = 0 \rightarrow [1]: 5.83 - 21\lambda_2 + 0.1\lambda_3 = 0 \\ [2]: -57.48 + 0.1\lambda_2 - 25\lambda_3 = 0 \end{array}\right\} \lambda_2 = 0.2666; \lambda_3 = -2.3 \Rightarrow NO VALIDO$$

Other hypothesis:

$$\left.\begin{array}{l}\lambda_3 \neq 0 \rightarrow [5]: p_A = 250p_B \\ \lambda_1 \neq 0 \rightarrow [3]: -20.8p_A - 49.9p_B + 110 = 0\end{array}\right\} p_A = 5.238; p_B = 0.0209$$

$$\left.\begin{array}{r}\lambda_2 = 0 \rightarrow [1]: 155.83 - 20.8\lambda_1 + 0.1\lambda_3 = 0 \\ [2]: -57.48 - 49.9\lambda_1 - 25\lambda_3 = 0\end{array}\right\} \lambda_1 = 7.41; \lambda_3 = -17.02 \Rightarrow NOVALIDO$$

Other hypothesis:

$$\left.\begin{array}{l}\lambda_1 \neq 0 \rightarrow [3]: -20.8p_A - 49.9p_B + 110 = 0 \\ \lambda_2 \neq 0 \rightarrow [4]: -21p_A + 0.1p_B + 35 = 0\end{array}\right\} p_A = 1.6738; p_B = 1.5067$$

$$\left.\begin{array}{r}\lambda_3 = 0 \rightarrow [1]: 5.8283 - 20.8\lambda_1 - 21\lambda_2 = 0 \\ [2]: 17.5202 - 49.9\lambda_1 + 0.1\lambda_2 = 0\end{array}\right\} \lambda_1 = 0.3509; \lambda_2 = -0.07008 \Rightarrow NOVALIDO$$

We continue assuming a non-zero value on one of the three Lagrange multipliers:

$$\left.\begin{array}{l}\lambda_1 \neq 0 \rightarrow [3]: -20.8p_A - 49.9p_B + 110 = 0 \\ \lambda_2 = 0 \rightarrow [1]: 42p_A - 0.2p_B - 20.8\lambda_1 - 64.17 = 0 \\ \lambda_3 = 0 \rightarrow [2]: -0.2p_A + 50p_B - 49.9\lambda_1 - 57.48 = 0\end{array}\right\} p_A = 1.70281; p_B = 1.49462; \lambda_1 = 0.338889$$

This point is valid, so it is the global minimum of the mathematical problem and global maximum of the original problem.

The price of product A will be 1.70281 and of article B will be 1.49462 and the overall benefit will be 51.951.

SOLUTION. PROBLEM 4 PART A

$$\min(x-9/4)^2+(y-2)^2$$
$$x^2-y\le 0 \qquad \Rightarrow \lambda_1$$
$$x+y-6\le 0 \qquad \Rightarrow \lambda_2$$
$$-x\le 0 \qquad \Rightarrow \lambda_3$$
$$-y\le 0 \qquad \Rightarrow \lambda_4$$

$$Con\,las\,condiciones\,KKT:$$
$$\nabla z(x,y)+\sum \lambda_i \nabla g_i(x,y)=0$$
$$\lambda_i g_i(x,y)=0$$
$$\lambda_i \ge 0$$

$$\nabla z(x,y)=\begin{bmatrix}2(x-9/4)\\2(y-2)\end{bmatrix}$$
$$\nabla g_1(x,y)=\begin{bmatrix}2x\\-1\end{bmatrix};\nabla g_2(x,y)=\begin{bmatrix}1\\1\end{bmatrix};\nabla g_3(x,y)=\begin{bmatrix}-1\\0\end{bmatrix};\nabla g_4(x,y)=\begin{bmatrix}0\\-1\end{bmatrix}$$

$$2(x-9/4)+\lambda_1 2x+\lambda_2-\lambda_3=0$$
$$2(y-2)-\lambda_1+\lambda_2-\lambda_4=0$$
$$\lambda_1(x^2-y)=0$$
$$\lambda_2(x+y-6)=0$$
$$-x\lambda_3=0$$
$$-y\lambda_4=0$$

The solution of the system is shown below:

$\bullet \lambda_4 \neq 0 \Rightarrow y = 0$

$$\circ \begin{cases} \lambda_1 \neq 0 \Rightarrow x = 0 \rightarrow \left| \begin{array}{l} \lambda_3 \neq 0 \\ \lambda_2 = 0 \end{array} \right. \\ -\dfrac{9}{2} - \lambda_3 = 0 \rightarrow \lambda_3 = -\dfrac{9}{2} \quad NO\,PUEDE\,DARSE : \lambda_i \geq 0 \end{cases}$$

$$\circ \begin{cases} \lambda_1 = 0 \Rightarrow -4 + \lambda_2 - \lambda_4 = 0 \Rightarrow \lambda_2 = 4 + \lambda_4 \\ \circ\circ \left| \begin{array}{l} \lambda_2 = 0 \Rightarrow \lambda_4 = -4 \quad NO\,PUEDE\,DARSE : \lambda_i \geq 0 \\ \lambda_2 \neq 0 \Rightarrow \left| \begin{array}{l} x = 6 \Rightarrow \lambda_3 = 0 \\ \dfrac{15}{2} + \lambda_2 = 0 \Rightarrow \lambda_2 = -\dfrac{15}{2} \quad NO\,PUEDE\,DARSE : \lambda_i \geq 0 \end{array} \right. \end{array} \right. \end{cases}$$

$\bullet \lambda_4 = 0 \Rightarrow y \neq 0$

$$\circ \begin{cases} \lambda_3 \neq 0 \Rightarrow x = 0 \rightarrow \left| -y\lambda_1 = 0 \rightarrow \lambda_1 = 0 \right. \\ -\dfrac{9}{2} + \lambda_2 - \lambda_3 = 0 \\ \left. \begin{array}{l} 2(y-2) + \lambda_2 = 0 \Rightarrow \underline{\lambda_2 = 0, y = 2} \\ (y-6)\lambda_2 = 0 \Rightarrow \underline{\lambda_2 = 0, y = 6} \end{array} \right\rangle IMPOSIBLE \end{cases}$$

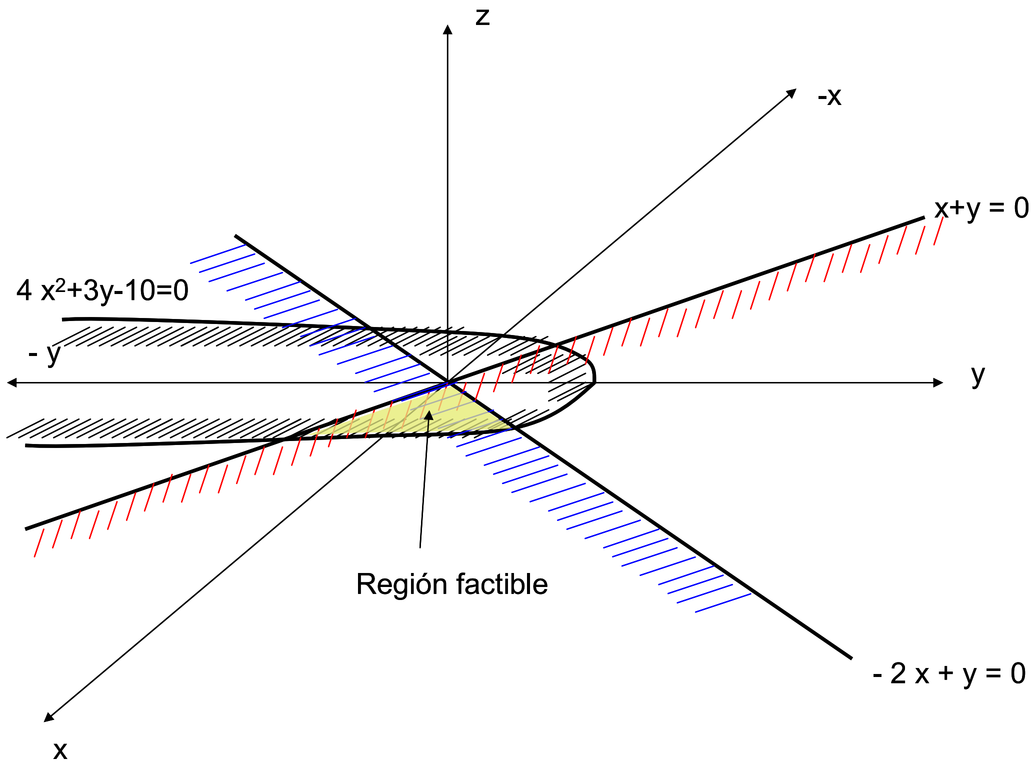
$$\circ \begin{cases} \lambda_3 = 0 \Rightarrow x \neq 0 \rightarrow \left| \begin{array}{l} \lambda_2 = 0 \\ 2(x - \dfrac{9}{4}) + \lambda_1 2x = 0 \\ 2(y-2) - \lambda_1 = 0 \end{array} \right. \\ \Rightarrow \left| \begin{array}{l} \lambda_1 \neq 0 \rightarrow y = x^2 \rightarrow \left| \begin{array}{l} 2x - \dfrac{9}{2} + \lambda_1 2x = 0 \\ 2x^2 - 4 - \lambda_1 = 0 \end{array} \right| \lambda_1 = 2x^2 - 4 \\ 2x - \dfrac{9}{2} + (2x^2 - 4)2x = 0 \\ 2x - \dfrac{9}{2} + 4x^3 - 8x = 0 \\ 4x^3 - 6x - \dfrac{9}{2} = 0 \Rightarrow \boxed{\begin{array}{l} x = 1.5 \\ y = 2.25 \end{array}} \end{array} \right. \end{cases}$$

$$\boxed{\begin{aligned} x &= 1.5 = \frac{3}{2} \\ y &= 2.25 = \frac{9}{4} \end{aligned}} \qquad \boxed{\begin{aligned} \lambda_1 &= 0.5 \\ \lambda_2 &= 0 \\ \lambda_3 &= 0 \\ \lambda_4 &= 0 \end{aligned}}$$

Minimum (global KKT): (3/2,9/4), multipliers (1/2,0,0,0)

Solution. Problem 4 part B

Sufficient conditions for a global maximum cannot be applied because the objective function is not convex, is concave. To check that these points are global optimums we have to verify that the constraints delimit the objective function. Then we have to find the points that satisfy Karush-Kuhn-Tucker necessary conditions and compare between them the value of the objective function.



This figure shows how the feasible region is bounded and therefore the objective function will also be bounded.

The necessary conditions of KKT are applied to a minimization problem with minor or equal constraints so the sign of the objective function and of the third constraint is changed. The problem to solve has the following formulation:

$$\begin{aligned}
&\min -\left(x-13/6\right)^2-\left(y-1/6\right)^2\\
&4x^2+3y-10\leq 0\\
&-2x+y\leq 0\\
&-x-y\leq 0
\end{aligned}$$

Since all of the constraint functions are differentiable the KKT necessary conditions can be applied for the existence of global minimums.

$$\nabla f(x^{*}) + \sum_i \lambda_i \nabla g_i\left(x^{*}\right) = 0$$
$$\lambda_i g_i\left(x^{*}\right) = 0; i = 1,\ldots,m$$
$$\lambda_i \geq 0; i = 1,\ldots,m$$

The KKT conditions of this problem are:

$$I)\ \text{-}\ 2\left(x\ \text{-}\ 13/6\right) + 8\lambda_1 x\ \text{-}\ 2\lambda_2\ \text{-}\ \lambda_3 = 0$$
$$II)\ \text{-}\ 2\left(y\ \text{-}\ 1/6\right) + 3\lambda_1 + \lambda_2\ \text{-}\ \lambda_3 = 0$$
$$III)\lambda_1\left(4x^2 + 3y\ \text{-}\ 10\right) = 0$$
$$IV)\lambda_2\left(\text{-}2x + y\right) = 0$$
$$V)\lambda_3\left(\text{-}x\ \text{-}\ y\right) = 0$$
$$VI)\lambda_1, \lambda_2, \lambda_3 \geq 0$$

Point A(1,2) is checked in expressions III), IV) and V) being λ_1, λ_2 and $\lambda_{3,} \neq 0$, $\neq 0$ and $= 0$ respectively.

Therefore, by substituting values of x=1, y=2 and λ_3=0 expressions I) and II) form a two equation system with two unknowns.

$$\left.\begin{aligned} 7/3 + 8\lambda_1 - 2\lambda_2 = 0 \\ -11/3 + 3\lambda_1 + \lambda_2 = 0 \end{aligned}\right\} \lambda_1 = \frac{5}{14}; \lambda_2 = \frac{109}{42}$$

Substituting in the objective function point A we obtain f(1,2)=85/18.

Point B(2,-2) is checked in expression III), IV) and V) being λ_1, λ_2 and $\lambda_{3,} \neq 0$, $= 0$ and $\neq 0$, respectively.

Therefore, by substituting values of x=2, y=-2 and λ_2=0 expressions I) and II) form two equation system with two unknowns.

$$\left.\begin{aligned} 2/6 + 16\lambda_1\ \text{-}\ \lambda_3 = 0 \\ 26/6 + 3\lambda_1\ \text{-}\ \lambda_3 = 0 \end{aligned}\right\} \lambda_1 = \frac{4}{13}; \lambda_3 = \frac{205}{39}$$

Substituting in the objective function point B we obtain f(2,-2)=85/18

Point C(0,0) is checked in the expressions III), IV) and V) being λ_1, λ_2 and $\lambda_{3,} = 0$, $\neq 0$, and $\neq 0$, respectively.

Therefore, by substituting the values of x=0, y= 0 and λ_1=0 expressions I) and II) form a two-equation system with two unknowns.

$$\left.\begin{aligned} 26/6 - 2\lambda_2 - \lambda_3 = 0 \\ 1/3 + \lambda_2 - \lambda_3 = 0 \end{aligned}\right\} \lambda_2 = \frac{4}{3}; \lambda_3 = \frac{5}{3}$$

Substituting in the objective function point C we obtain f(0,0)=85/18

The rest of the points that satisfy the KKT conditions (making two of the values of λ to be null) do not obtain higher values of the objective function. Therefore points A, B and C are global optimums.

$$\left.\begin{array}{l}\lambda_1 = 0\\ \lambda_2 = 0\\ \lambda_3 \neq 0\end{array}\right\} x = 1; y = -1; f(1,-1) = {}^{49}\!/_{18}$$

$$\left.\begin{array}{l}\lambda_1 = 0\\ \lambda_2 \neq 0\\ \lambda_3 = 0\end{array}\right\} x = {}^{1}\!/_{2}; y = 1; f({}^{1}\!/_{2},1) = {}^{125}\!/_{36}$$

<u>SOLUTION PROBLEM 4 PART C</u>

Lagrangian:

$$L(x,y,z,\lambda,\mu) = -3x + y - z^2 + \lambda(x+y+z) + \mu\left(-x+2y+z^2\right)$$

Necessary KKT conditions for a local minimum:

$$\begin{aligned}
&\textit{Punto factible}\\
&\textit{Funciones } f, g \text{ } y \text{ } h \textit{ diferenciables en } x^*\\
&\nabla g(x^*) \, y \, \nabla h(x^*) \textit{ linealmente independientes}\\
&\nabla f(x^*) + \lambda \nabla g(x^*) + \mu \nabla h(x^*) = 0\\
&\lambda g(x^*) = 0\\
&\lambda \geq 0
\end{aligned}$$

These conditions are:

$$\begin{aligned}
&-3 + \lambda - \mu = 0 \quad [1]\\
&1 + \lambda + 2\mu = 0 \quad [2]\\
&-2z + \lambda + 2\mu z = 0 \quad [3]\\
&\lambda(x+y+z) = 0 \quad [4]\\
&-x + 2y + z^2 = 0 \quad [5](\textit{punto factible})\\
&\lambda \geq 0
\end{aligned}$$

- From [1] and [2] we obtain that $\lambda = \frac{5}{3}$ and $\mu = -\frac{4}{3}$
- Knowing the value of λ you can calculate the value of z in equation [3], in which the value of z is equal to $\frac{5}{14}$.
- As $\lambda \neq 0$ from [4] we obtain $x+y+z=0$
- Substituting the value of z in equation [5] and taking into account the previous expression results:

$$\left.\begin{aligned}
&x + y + \frac{5}{14} = 0\\
&-x + 2y + \frac{25}{196} = 0
\end{aligned}\right\} \Rightarrow x = -\frac{115}{588}; y = -\frac{95}{588}$$

The candidate point is local minimum but is <u>not a global minimum</u> due to several reasons:

1) The objective function is concave even though the constraints are convex

2) The value of $\mu < 0$ and constraint h is convex (one of the two circumstances should be different)

SOLUTION TRIATHLON

The objective is turned into the following nonlinear optimization model

$$\begin{aligned}
&\min \frac{4000}{x_1} + 90 - \left(\frac{x_2}{50}\right)^2 + \sqrt{1600 - x_3} \\
&100 \le x_1 \le 1000 \\
&100 \le x_2 \le 400 \\
&400 \le x_3 \le x_1 \\
&x_1 + x_2 + x_3 \le 1600
\end{aligned}$$

whose associate Lagrange function is

$$\begin{aligned}
&L(x_1, x_2, x_3; \lambda_1, \ldots\ldots, \lambda_7) = \frac{4000}{x_1} + 90 - \left(\frac{x_2}{50}\right)^2 + \sqrt{1600 - x_3} + \\
&\lambda_1 (x_1 + x_2 + x_3 - 1600) + \lambda_2 (100 - x_1) + \lambda_3 (x_1 - 1000) + \\
&\lambda_4 (100 - x_2) + \lambda_5 (x_2 - 400) + \lambda_6 (400 - x_3) + \lambda_7 (x_3 - x_1)
\end{aligned}$$

For this model the necessary Karush-Kuhn-Tucker conditions for an optimum are:

$$
\begin{aligned}
&\frac{\partial L}{\partial x_1} = -\frac{4000}{x_1^{\ 2}} + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_7 = 0 \\
&\frac{\partial L}{\partial x_2} = -\frac{2x_2}{2500} + \lambda_1 - \lambda_4 + \lambda_5 = 0 \\
&\frac{\partial L}{\partial x_3} = -\frac{1}{2\sqrt{1600 - x_3}} + \lambda_1 - \lambda_6 + \lambda_7 = 0 \\
&\lambda_1 \left(x_1 + x_2 + x_3 - 1600\right) = 0 \\
&\lambda_2 \left(100 - x_1\right) = 0 \\
&\lambda_3 \left(x_1 - 1000\right) = 0 \\
&\lambda_4 \left(100 - x_2\right) = 0 \\
&\lambda_5 \left(x_2 - 400\right) = 0 \\
&\lambda_6 \left(400 - x_3\right) = 0 \\
&\lambda_7 \left(x_3 - x_1\right) = 0 \\
&100 \le x_1 \le 1000 \\
&100 \le x_2 \le 400 \\
&400 \le x_3 \le x_1 \\
&x_1 + x_2 + x_3 \le 1600 \\
&\lambda_i \ge 0, i = 1, \ldots., 7
\end{aligned}
$$

The proposed point, , $x_1 = 700$, $x_2 = 400$, $x_3 = 500$, make all the parameters null excepts λ_1 and λ_5. Taking the first and third KKT constraints we obtain two different values for λ_1. Therefore, this point is not an optimum.

SOLUTION. CUBIC FUNCTION

a) Establish the sufficient KKT conditions and, if necessary, the sufficient and necessary conditions

$$\nabla^2(x_1,x_2)=\begin{bmatrix}2 & 0\\ 0 & 6x_2\end{bmatrix}$$

The objective function has a positive definite hessian in the feasible region $\left[x_1\geq 0; x_2\geq 0\right]$ so this region is convex. Moreover constraints are also convex so necessary and sufficient KKT conditions that the global optimum of the problem satisfies can be applied. These conditions are established from the Lagrangian and the constraints of the problem.

$$L(x_1,x_2)=x_1^2-6x_1+x_2^3-3x_2+\lambda_1(x_1+x_2-1)+\lambda_2(-x_1)+\lambda_3(-x_2)$$

$$\begin{array}{ll}
[1] & \nabla L\Big|_{x_1}=2x_1-6\quad+\lambda_1-\lambda_2=0\\
[2] & \nabla L\Big|_{x_2}=3x_2^2-3\quad+\lambda_1-\lambda_3=0\\
[3] & \lambda_1(x_1+x_2-1)=0\\
[4] & \lambda_2\quad(-x_1)=0\\
[5] & \lambda_3\quad(-x_2)=0
\end{array}$$

$$\left.\begin{array}{ll}
[6] & x_1+x_2-1\leq 0\\
[7] & -\quad x_1\leq 0\\
[8] & -\quad x_2\leq 0
\end{array}\right\}\textit{Punto}\quad\textit{factible}$$

b) Calculate a minimum, if possible, by the fulfillment of the KKT conditions

- Assuming $\lambda_3\neq 0$ from [5] we obtain $x_2=0$
- Assuming $\lambda_1\neq 0$ from [3] we obtain $x_1=1$
- From [4] we obtain $\lambda_2=0$
- From [1] we obtain $\lambda_1=4$
- From [2] we obtain $\lambda_3=1$

The obtained point (1, 0) has all Lagrange multipliers greater o equal to zero. When the necessary and sufficient conditions are fulfilled this point is a global minimum of the problem in the feasible region defined by the constraints.

c) Indicate what type of point is $\left(\frac{1}{2},\frac{1}{2}\right)$

This point fulfills the constraints, so it is a feasible point. When KKT conditions are applied in this point it is checked that conditions are not fulfill so this point is not important.

- From [4] $\lambda_2=0$

- From [5] $\lambda_3 = 0$
- From [1] $\lambda_1 = \dfrac{9}{4}$
- From [2] $\lambda_1 = 5$. So KKT conditions are not fulfilled.

<u>Solution. Science of complexity</u>

Being:

- x: hours spent studding TF. It has to satisfy $5 \le 2.5\sqrt{x} \le 10 \Rightarrow 4 \le x \le 16$
- y: hours spent studding TA, satisfying $5 \le \dfrac{5}{1.5}\log_4 y \le 10 \Rightarrow 8 \le y \le 64$

  We must solve the following nonlinear optimization model

$$\begin{aligned}
&\max 2.5\sqrt{x} + \frac{5}{1.5}\log_4 y \\
&x + y \le 20 \\
&x \ge 4, x \le 16 \\
&y \ge 8, y \le 64
\end{aligned}$$

who has the following Lagrange function:

$$\begin{aligned}
L(x,y,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5) &= 2.5\sqrt{x} + \frac{5}{1.5}\log_4 y + \lambda_1(20-x-y) + \\
&+ \lambda_2(x-4) + \lambda_3(16-x) + \lambda_4(y-8) + \lambda_5(64-y)
\end{aligned}$$

The Karush-Kuhn-Tucker conditions of this model are:

$$\begin{aligned}
&\frac{2.5}{2}\frac{1}{\sqrt{x}} - \lambda_1 + \lambda_2 - \lambda_3 = 0 \\
&\frac{5}{1.5}\frac{1}{\ln 4}\frac{1}{y} - \lambda_1 + \lambda_4 - \lambda_5 = 0 \\
&(a)\,\lambda_1(20-x-y) = 0 \\
&(b)\,\lambda_2(x-4) = 0 \\
&(c)\,\lambda_3(16-x) = 0 \\
&(d)\,\lambda_4(y-8) = 0 \\
&(e)\,\lambda_5(64-y) = 0 \\
&\lambda_i \ge 0, i = 1,2,3,4,5
\end{aligned}$$

and the initial group [1] of the constraints of the model:

Si $x = 12$ e $y = 8 \Rightarrow \lambda_1 \ge 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 \ge 0, \lambda_5 = 0$

$$\left.\begin{aligned}
\frac{2.5}{2}\frac{1}{\sqrt{12}} - \lambda_1 = 0 \\
\frac{5}{1.5}\frac{1}{\ln 4}\frac{1}{8} - \lambda_1 + \lambda_4 = 0
\end{aligned}\right\} \Rightarrow
\begin{aligned}
\lambda_1 &= \frac{2.5}{4\sqrt{3}} > 0 \\
\lambda_4 &= \frac{2.5}{4\sqrt{3}} - \frac{5}{12\ln 4} = 5\left(\frac{1}{8\sqrt{3}} - \frac{1}{12\ln 4}\right) > 0
\end{aligned}$$

So, the suggested point satisfies the necessary conditions of side. The hessian of the objective function is

$$H = \begin{bmatrix} -\dfrac{2.5}{4}x^{-3/2} & 0 \\ 0 & -\dfrac{5}{1.5}\dfrac{1}{\ln 4}\dfrac{1}{y^2} \end{bmatrix}$$

For every point of the constraints domain the matrix is a negative definite quadratic form. This means that the objective function is concave in such domain. Moreover, the group of constraints as they are all lineal constraints is convex.

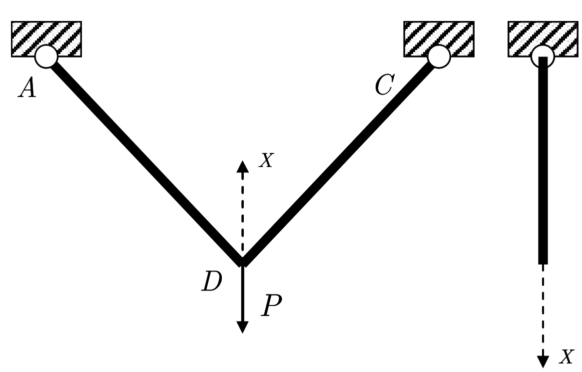
For this reason, the solution is $x = 12$, $y = 8$ and represents a global maximum.

<u>SOLUTION: SIMPLE STRUCTURE</u>

The objective function of the problem is the minimization of the structure's volume as density and price are constant for the same type of material. The objective function is:

$$\min\left(\frac{2L}{\cos\alpha} + L\right)\cdot A$$

To solve the problem it is necessary to have some previous knowledge in mechanical structures. This problem is solved dividing the hyperstatic structure into two structures that later will be related between each other and are shown in the following figure. Deformations of both structures on the edge are equal when a vertical and upwards stress X is applied on the first structure. This stress is unknown and is applied by the central rod of the ensemble of both rods, along with the earlier stress P. The second structure on the right the only stress applied is vertical stress X but downwards.



The structure's deformation on the left, being E the modulus of elasticity of the steel, is:

$$2\left[-\frac{1}{2\cos\alpha}\cdot\frac{(P-X)\dfrac{1}{2\cos\alpha}}{AE}\frac{L}{\cos\alpha}\right]=\frac{L}{AE}\left[\frac{P-X}{2\cos\alpha}\right]$$

The structure's deformation of the right is:

$$\frac{XL}{AE}$$

Equaling both expressions we can obtain stress X as:

$$X=\frac{P}{1+2\cos\alpha}$$

The stresses on the rods have to be lower than σ_{adm} and therefore for the rods joined in angle the resultant stress on the rod is calculated as the sum of the effect of P and X. This effect has to be lower or equal to the multiplication of the allowable stress by the section of the rod as it is indicated below:

$$\frac{P-X}{2\cos\alpha}\le\sigma_{adm}A$$

Substituting the expression of X in the previous inequation turns into the following constraint:

$$A(1+2\cos\alpha)\ge\frac{P}{\sigma_{adm}}$$

Applying the previous principle to the vertical rod and the stress X applied in the vertical axes and downwards we obtain the same constraint as the previous one with the two rods in angle. We can check applying the same principle to the following expression.

$$X\le\sigma_{adm}A$$

Furthermore the distance between point A and B is equal to d and therefore the length of the sloping rod and the angle are related as follows:

$$Lsen\alpha=d$$

Therefore, variable L can be obtained relating it with the angle $\cos\alpha$ and applying $sen^2\alpha+\cos^2\alpha=1$ we obtain:

$$L = \frac{d}{\sqrt{1-\cos^2 \alpha}}$$

Designating $\cos\alpha$ as variable Z and limiting its value between 0 and 1 we obtain the following nonlinear optimization problem when we replace L with its expression with Z.

$$\begin{aligned}
&\min\left(\frac{2d}{\sqrt{1-Z^2}Z}+\frac{d}{\sqrt{1-Z^2}}\right)A \\
&sujeto\, a: \\
&A(1+2Z) \geq \frac{P}{\sigma_{adm}} \\
&0 \leq Z \leq 1 \\
&A \geq 0
\end{aligned}$$

The numerical formulation results in the following expression being the section expressed in quadratic centimeters and not having the right term of the constraint raised as it shown below:

$$\begin{aligned}
&\min\left(\frac{6}{\sqrt{1-Z^2}Z}+\frac{3}{\sqrt{1-Z^2}}\right)\frac{A}{10^4} \\
&sujeto\, a: \\
&A(1+2Z) \geq \frac{300}{235} \\
&0 \leq Z \leq 1 \\
&A \geq 0
\end{aligned}$$

The optimization result has been:

$$\begin{aligned}
&A = 0.509\, cm^2 \\
&Z = 0.755 \Rightarrow \alpha = \cos^{-1}(0.755) = 45.5^o \\
&L = \frac{3}{\sqrt{1-Z^2}} = 4.57 metros
\end{aligned}$$

The resulting volume is 8.4909 10^{-4} m^3 [a] and therefore the price to pay is 6.19 €

```
VARIABLES
Z   coseno del ángulo entre una barra inclinada y la vertical
A   sección de la barra en centimetros cuadrados
VOL volumen de la estructura

POSITIVE VARIABLES L,Z,A;
```

```
Z.UP=1;
Z.LO=0.01;

EQUATIONS
FOVOL volumen de la estructura
LIM   igualdad de la deformación en las dos estructuras complementarias ;

FOVOL.. VOL =E=[(6/Z)+3]*A*0.0001/SQRT(1-Z*Z);
LIM.. A*(1+2*Z) =G= (300/235);

MODEL ESTRUCTURA /ALL/

SOLVE ESTRUCTURA MINIMIZING VOL USING NLP
```