PROBLEM: MOBILE PHONES

A new mobile phone provider, TUMOVIL, would like to enter the Spanish market by stealing some clients from other competing companies such as MOVILETE, which has 100000 (hundred thousand) clients.

TUMOVIL is going to launch three different types of tariffs: TuTrabajo (TT), TusAmigos (TA), and TuCiudad (TC).

TUMOVIL spends a budget of 2 million euros on advertisements. Their market studies department has evaluated the impact of a very aggressive advertisement strategy on MOVILETE's clients. Supposing that they can only spend the full million amounts (and no fractions) on publicity, these are the conclusions:

For each million spent on advertising TT, 10,000 clients leave MOVILETE and go to TUMOVIL. However, the maximum number of clients that can be stolen from MOVILETE using this type of publicity is 10,000.

For each million spent on advertising TA, 7,500 clients are stolen from MOVILETE. The maximum number of clients that can be stolen from MOVILETE using this type of publicity is 15,000.

For each million used in advertising TC, 5,000 clients leave MOVILETE. The maximum number of clients stolen from MOVILETE using solely this advertisement strategy is 10,000.

MOVILETE, on the other hand, only spends 1 million euros on advertisements. Their market analysis has yielded the following result: when they invert 1 million in counter-publicity against one specific tariff of the three tariffs of TUMOVIL, then this neutralizes the loss of the same number of clients that 1 million euros of TUMOVIL's advertisement for this tariff would have caused.

Please carry out the following tasks:

a) *Derive the payoff matrix (or table).*
b) *Does a dominated strategy exist? If the answer is yes, indicate which one/ones are the dominated strategies.*
c) *Taking the point of view of MOVILETE and supposing that the non-dominated strategies of TUMOVIL are equally likely, point out the decisions of MOVILETE under the following criteria:*

   c.1) Under the Wald criterion (MAXIMIN) *c.2) Under the Savage criterion (MINIMAX)*

c.3) Under the Laplace criterion

d) Do one or multiple Nash equilibrium points exist? If it does not exist, derive a mixed strategy.

PROBLEM: LITTLE RED RIDING HOOD

The sweet, fragile, and blonde little Red Riding Hood visits her poor, sick grandmother daily and brings her a box of 50 pies, which are very likely the cause of Granny's diabetes. But on her way to granny's house, she runs into "the wolf", which is the nickname of an individual very well-known for ferociously devouring all types of pies, who specializes in taking pies off of young ladies between 10 and 30 years who enter the forest.

To avoid such an undesirable encounter, Little Red Riding Hood can go to Granny's house via the mountain path or the easier straight path, and if she so chooses, she can convince her boyfriend to accompany her on each of the two paths.

"The wolf" can do the same (obviously, we are talking about the two paths options, not the boyfriend) and can walk or go by bike (risky business for his teeth if he chooses the mountain path). The table below contains the pies that "the wolf" can snatch off Little Red Riding Hood under all possible circumstances. The negative quantity in the table corresponds to what happens when "the wolf" has the bad luck to encounter Little Red Riding Hood's boyfriend, who visits the gym regularly and weighs more than 120kg consisting of pure muscle, that is that "the wolf" can expect to be beaten up very badly and on top of everything the boyfriend will even take away his backpack.

	Little Red Riding Hood			
"The wolf"	**Alone on the easy path**	**Alone on the mountain path**	**With boyfriend on the easy path**	**With boyfriend on the mountain path**
By bike on the easy path	*10*	*0*	*5*	*0*
By bike on the mountain path	*0*	*5*	*0*	*40*
By foot on the easy path	*30*	*0*	*-20*	*0*

By foot on the mountain path	*0*	*15*	*0*	*50*

a) How do Little Red Riding Hood and "the Wolf" have to choose their paths to end up at an equilibrium point?

b) How many pies does "the wolf" expect to get per encounter?

PROBLEM: ZERO-SUM GAME 3 X 4

Two players, J1 and J2, face a rectangular game whose payoff matrix is shown below, representing what J2 has to pay to J1.

		J2			
		I	**II**	**III**	**IV**
J1	**A**	*6*	*4*	*5*	*6*
	B	*3*	*6*	*4*	*8*
	C	*5*	*3*	*4*	*4*

The payments are expressed in units of 10000 €.

a) *Show that both J1 and J2 have one dominated strategy.*

b) *Solve the game graphically, taking the point of view of player J1.*

c) *Considering the obtained result, demonstrate that J2 never uses strategy I.*

d) *Show that the optimal strategy of J2 is*

$$Y^* = (y_1, y_2, y_3, y_4) = \left(0, \frac{1}{3}, \frac{2}{3}, 0\right)$$

e) *A spy offers J1 to tell him/her what J2's strategy will be beforehand. What is the maximum amount that J1 should be willing to pay the spy for this deal to remain profitable for J1?*

PROBLEM: ZERO-SUM GAME 2 X 4

Solve the game whose payoff matrix towards player J1 is shown below

	J2			
J1	8	9	3	5
	2	4	7	5

PROBLEM: ZERO-SUM GAME 3 X 5

Consider the following rectangular game:

		J2				
		A	**B**	**C**	**D**	**E**
J1	**1**	5	8	8	7	9
	2	8	6	4	5	6
	3	6	8	5	6	7

a) *Eliminate the dominated strategies and check that the game can be reduced to a 2x3 game.*

b) *Solve the game graphically from the point of view of player J1*

c) *Set up the linear programming model of J2 and starting from the solution of player J1, obtain the solution of player J2*

SOLUTION: MOBILE PHONES

a) Derive the payoff matrix

Since TUMOVIL disposes of 2 million euros for advertisements, let us now describe the possible pure strategies of this company, which can be obtained by dividing the 2 million among the three different tariffs (TT, TA, TC). Note that the numbers in parenthesis correspond to the millions invested in the advertisement for that tariff if and only if this investment increases the number of clients for the company.

The values of these strategies are (1,1,0), (0,2,0), (1,0,1), (0,1,1), (0,0,2), and (2,0,0), which correspond to 17500, 15000, 15000, 12500, 10000 and 10000 clients that have been stolen from MOVILETE.

The strategies of company MOVILETE are to advertise against TT, against TA, or against TC. If MOVILETE advertises against TT, then they do not lose 10000 clients; if they advertise against TA, they maintain 7500 clients, and advertising against TC maintains 5000 clients.

The arising payoff matrix is:

		TUMOVIL					
		(1,1,0)	(0,2,0)	(1,0,1)	(0,1,1)	(0,0,2)	(2,0,0)
MOVILETE	TT	7500	15000	5000	12500	10000	10000
	TA	10000	7500	15000	5000	10000	10000
	TC	17500	15000	10000	7500	5000	10000

b) Does a dominated strategy exist? If the answer is yes, indicate which one/ones are the dominated strategies.

		TUMOVIL					
		(1,1,0)	(0,2,0)	(1,0,1)	~~(0,1,1)~~	~~(0,0,2)~~	(2,0,0)
MOVILETE	TT	7500	15000	5000	~~12500~~	~~10000~~	10000
	TA	10000	7500	15000	~~5000~~	~~10000~~	10000
	TC	~~17500~~	~~15000~~	~~10000~~	~~7500~~	~~5000~~	~~10000~~

The strategy (0,1,1) of TUMOVIL is dominated by (0,2,0), and strategy (0,0,2) is dominated by (2,0,0). TT dominates strategy TC of MOVILETE.

		TUMOVIL			
		~~(1,1,0)~~	(0,2,0)	(1,0,1)	(2,0,0)
MOVILETE	TT	~~7500~~	15000	5000	10000
	TA	~~10000~~	7500	15000	10000

Strategy (1,1,0) of TUMOVIL is dominated by (2,0,0)

		TUMOVIL		
		(0,2,0)	(1,0,1)	~~(2,0,0)~~
MOVILETE	TT	15000	5000	~~10000~~
	TA	7500	15000	~~10000~~

Strategy (2,0,0) is dominated by the combination of 2/3 of strategy (0,2,0) and 1/3 of strategy (1,0,1)

The resulting payoff matrix (after having eliminated all dominated strategies) is:

		TUMOVIL	
		(0,2,0)	(1,0,1)
MOVILETE	TT	15000	5000
	TA	7500	15000

c) Taking the point of view of MOVILETE and supposing that the non-dominated strategies of TUMOVIL are equally likely, point out the decisions of MOVILETE under the following criteria:

c.1) Under the Wald criterion (MAXIMIN)

The Wald criterion establishes its decision (or chosen strategy) as follows: if the worst happens for each decision (which means that the other player chooses the strategy that is worst for us in each case), then we choose the strategy that yields the best result under all these worst cases.

		TUMOVIL		
		(0,2,0)	(1,0,1)	Maximum
MOVILETE	TT	15000	5000	15000
	TA	7500	15000	15000

From the point of view of MOVILETE, the worst that could happen for strategy TT is that we lose 15000 clients (which happens when TUMOVIL chooses (0,2,0)), and the worst that could happen for TA is that we lose 15000 clients (when TUMOVIL chooses (1,0,1)). The worst case for each strategy of TT is indicated in the last column. Considering the worst-case scenario for each strategy, MOVILETE wants to choose the one that is best among all the worst cases. Since both strategies lead to the same result, both decisions can be made under the Wald criterion.

c.2) Under the Savage criterion (MINIMAX)

The opportunity loss or regret matrix is therefore:

		TUMOVIL		
		(0,2,0)	(1,0,1)	Maximum
MOVILETE	TT	7500	0	7500
	TA	0	10000	10000

The minimum regret is obtained through the advertising strategy against TT.

c.3) Under the Laplace criterion

		TUMOVIL		
		(0,2,0)	(1,0,1)	Medium value
MOVILETE	TT	15000	5000	10000
	TA	7500	15000	11250

Therefore, we choose the strategy that yields the smaller medium value, i.e., strategy TT.

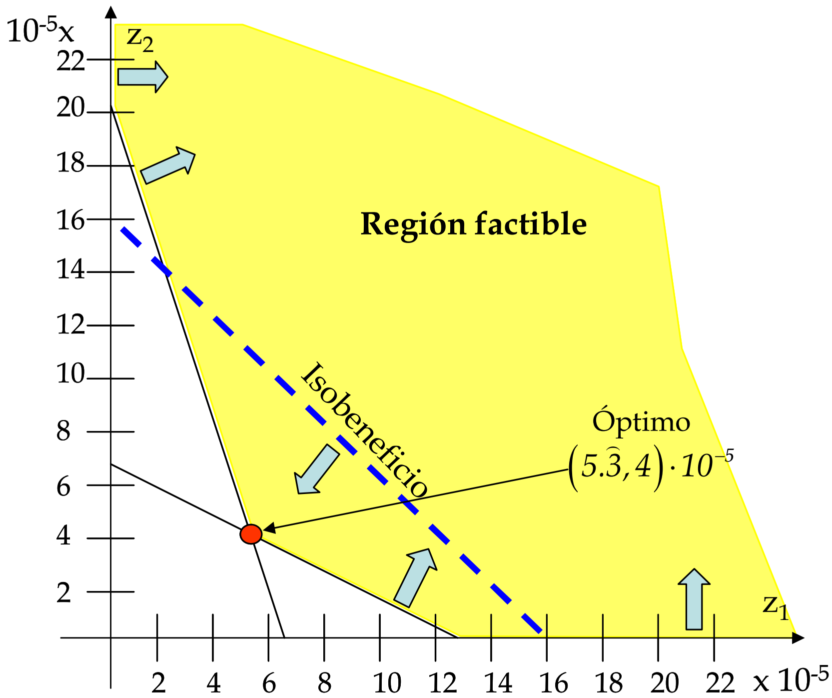
d) Do one or multiple Nash equilibrium points exist? If it does not exist, derive a mixed strategy.

		TUMOVIL		
		(0,2,0)	(1,0,1)	*Maximum*
MOVILETE	TT	15000	5000	15000
	TA	7500	15000	15000
Minimum		7500	5000	

Since the minimum per column does not coincide with the máximum per row, there is no Nash equilibrium in pure strategies, and therefore, the equilibrium has to be obtained in mixed strategies.

$$\left.\begin{array}{l} max\, v \\ v \leq 15000y_1 + 5000y_2 \\ v \leq 7500y_1 + 15000y_2 \\ y_1 + y_2 = 1 \\ y_1, y_2 \geq 0 \end{array}\right\} dividing\, by\, v \begin{cases} min\, z_1 + z_2 \\ 15000z_1 + 5000z_2 \geq 1 \\ 7500z_1 + 15000z_2 \geq 1 \\ z_1, z_2 \geq 0 \end{cases}$$

Solving this problem graphically yields:



The graphical solution yields that the optimal solution for (z_1, z_2) *has the values* $\left(5.\widehat{3}, 4\right) \cdot 10^{-5}$ *representing the red point in the figure above. Undoing the transformation* *of the variable, we obtain the value of the probabilities of each pure strategy at equilibrium for the player of the columns (i.e., TUMOVIL), which are:*

$$v = \frac{1}{z_1 + z_2} = \frac{1}{9.\widehat{3} \cdot 10^{-5}} = 10714.28$$
$$y_1 = z_1 v = 0.5714$$
$$y_2 = z_2 v = 0.4286$$

The values of the probabilities at equilibrium for the player of the rows (i.e., MOVILETE) are the solution to the dual problem of the previous problem (which was the problem from the point of view of TUMOVIL). Applying our linear programming knowledge, the solution of the dual can be obtained from the value of the relative cost coefficients of the slack variables of the primal problem.

The dual problem has the following formulation, to which we also apply the variable transformation (dividing by the value of the game v).

$$\left.\begin{array}{l} \min v \\ v \geq 15000x_1 + 7500x_2 \\ v \geq 5000x_1 + 15000x_2 \\ x_1 + x_2 = 1 \\ x_1, x_2 \geq 0 \end{array}\right\} \begin{array}{c} w_1 = \dfrac{x_1}{v} \\ w_2 = \dfrac{x_2}{v} \\ \text{dividing by } v \end{array} \left\{\begin{array}{l} \max w_1 + w_2 \\ 15000w_1 + 5000w_2 \leq 1 \\ 7500w_1 + 15000w_2 \leq 1 \\ w_1, w_2 \geq 0 \end{array}\right.$$

After the variable transformation, the primal problem is formulated in standard form (minimization). Given that the basic variables are z_1 and z_2, the nonbasic variables are the surplus variables e_1 and e_2.

$$\begin{array}{l} \min z_1 + z_2 \\ 15000z_1 + 5000z_2 - e_1 \qquad = 1 \\ 7500z_1 + 15000z_2 \qquad - e_2 = 1 \\ z_1, z_2 \geq 0 \end{array}$$

The relative cost coefficients of the nonbasic variables can be calculated as

$$\hat{C}_N^T = C_N^T - C_B^T B^{-1} N = (0,0) - (1,1)\begin{pmatrix} 15000 & 5000 \\ 7500 & 15000 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \left(4, 5.\widehat{3}\right)10^{-5}$$

Therefore, the value of the variables w_1 and w_2 is the value of the reduced cost of the excess variables e_1 and e_2 that are previously computed. The value of the probabilities of the column player, x_1 and x_2 are obtained by undoing the variable transformation and are therefore:

$$x_1 = w_1 v = 0.4286$$
$$x_2 = w_2 v = 0.5714$$

SOLUTION: LITTLE RED RIDING HOOD

a) How do Little Red Riding Hood and "the Wolf" have to choose their paths to end up at an equilibrium point?

First of all, we simplify the payoff matrix eliminating dominated strategies, taking into account that the quantities of the table represent the payments made to the row player ("the wolf").

The strategy "Alone on the easy path" is dominated by the strategy "With boyfriend on the easy path." The strategy "With boyfriend on mountain path" is dominated by "Alone on mountain path".

	Little Red Riding Hood			
"The wolf"	**Alone on the easy path**	**Alone on the mountain path**	**With boyfriend on the easy path**	**With boyfriend on the mountain path**
By bike on easy path	*10*	*0*	*5*	*0*
By bike on the mountain path	*0*	*5*	*0*	*40*
By foot on easy path	*30*	*0*	*-20*	*0*
By foot on the mountain path	*0*	*15*	*0*	*50*

Now, let us eliminate the dominated strategies of the Wolf. The decision "By foot on easy path" is dominated by the decision "by bike on easy path". Moreover, we eliminate the strategy "By bike on mountain path" since it is dominated by "By foot on mountain path", as seen in the following table.

	Alone on the mountain path	**With boyfriend on the easy path**

By bike on the easy path	*0*	*5*
By bike on the mountain path	*5*	*0*
By foot on the easy path	*0*	*-20*
By foot on the mountain path	*15*	*0*

Hence, the table with the simplified decisions is:

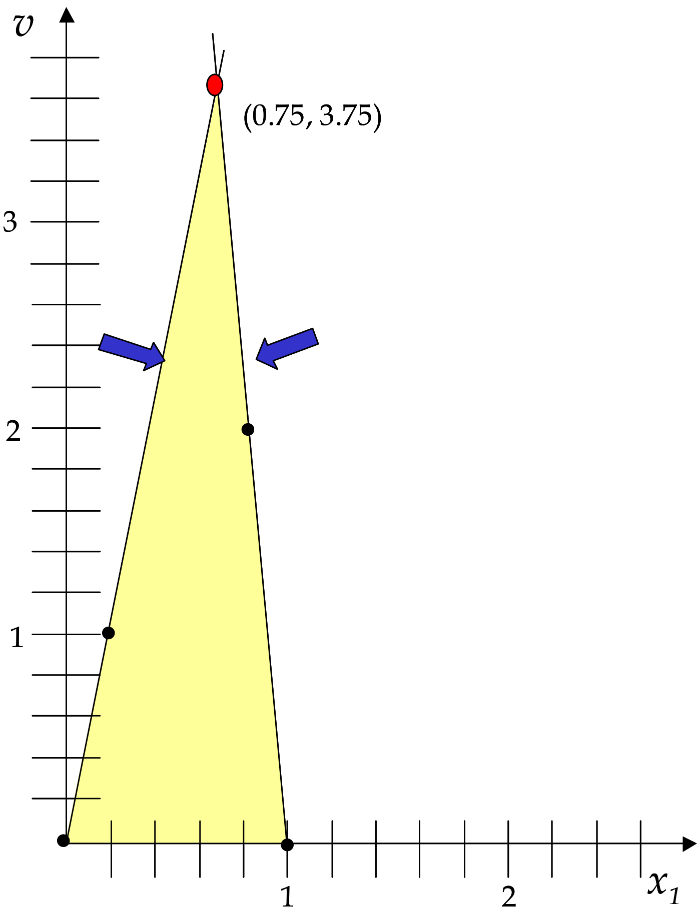
	Alone on the mountain path	**With boyfriend on the easy path**
By bike on the easy path	*0*	*5*
By foot on the mountain path	*15*	*0*

In this game, there is no equilibrium in pure strategies, and therefore, the equilibrium in mixed strategies has to be obtained. Therefore, one has to set up the linear programming problem representing this game.

From the point of view of the Wolf, this game can be written as:

$$\left.\begin{array}{l} \max v \\ v \leq 15x_2 \\ v \leq 5x_1 \\ x_1 + x_2 = 1 \\ x_1, x_2 \geq 0 \end{array}\right\} \Rightarrow x_2 = 1 - x_1 \Rightarrow \begin{cases} \max v \\ v + 15x_1 \leq 15 \\ v - 5x_1 \leq 0 \\ v, x_1 \geq 0 \end{cases}$$

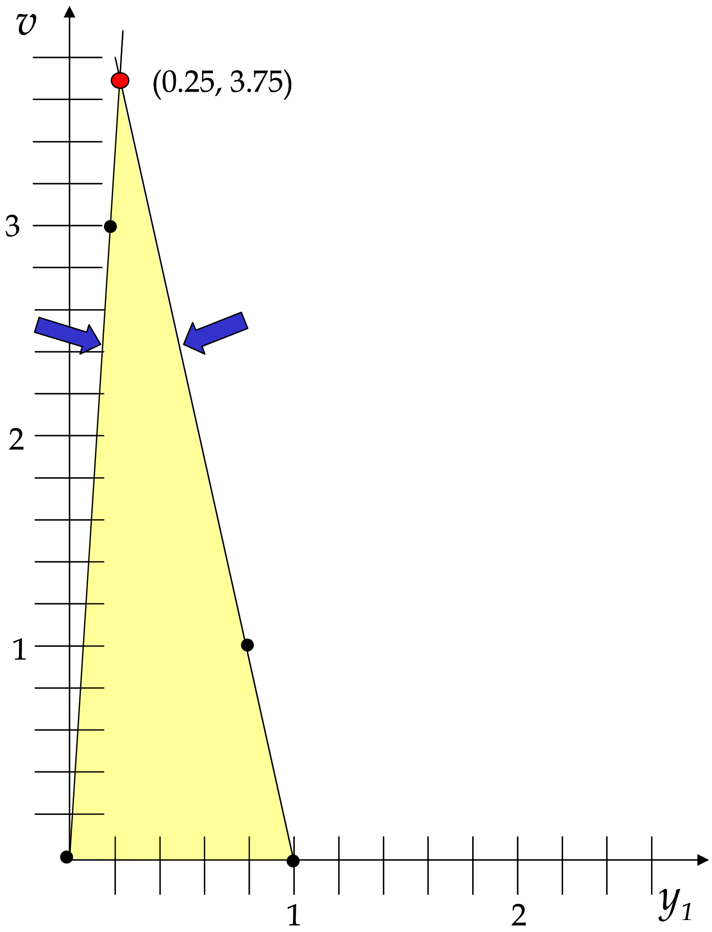
Solving this problem graphically yields:



Therefore, the strategies of the player "the Wolf" are chosen with probabilities $x_1 = 0.75$ *and* $x_2 = 0.25$ *and the value of the game is 3.75 (which is the number of pies obtained by the Wolf per encounter). The optimal decisions of player Little Red Riding Hood can be obtained by solving the following linear programming problem:*

$$\left.\begin{array}{l} \min v \\ v \geq 5y_2 \\ v \geq 15y_1 \\ y_1 + y_2 = 1 \\ y_1, y_2 \geq 0 \end{array}\right\} \Rightarrow y_2 = 1 - y_1 \Rightarrow \left\{\begin{array}{l} \max v \\ v + 5y_1 \geq 5 \\ v - 15y_1 \geq 0 \\ v, y_1 \geq 0 \end{array}\right.$$

Solving graphically yields:



Hence, the probabilities of the strategies of Little Red Riding Hood are $y_1 = 0.25$ *and* $y_2 = 0.75$ *and the value of the game is still 3.75 pies. Note that the value of the game of both problems has to be the same.*

Therefore, the equilibrium is obtained when three out of four times "the Wolf" goes "by bike on the easy path," and once out of four times he goes "by foot on the mountain path." Little Red Riding Hood has to go "with their boyfriend on the easy path" three out of four times, and one out of four times, she has to go "alone on the mountain path".

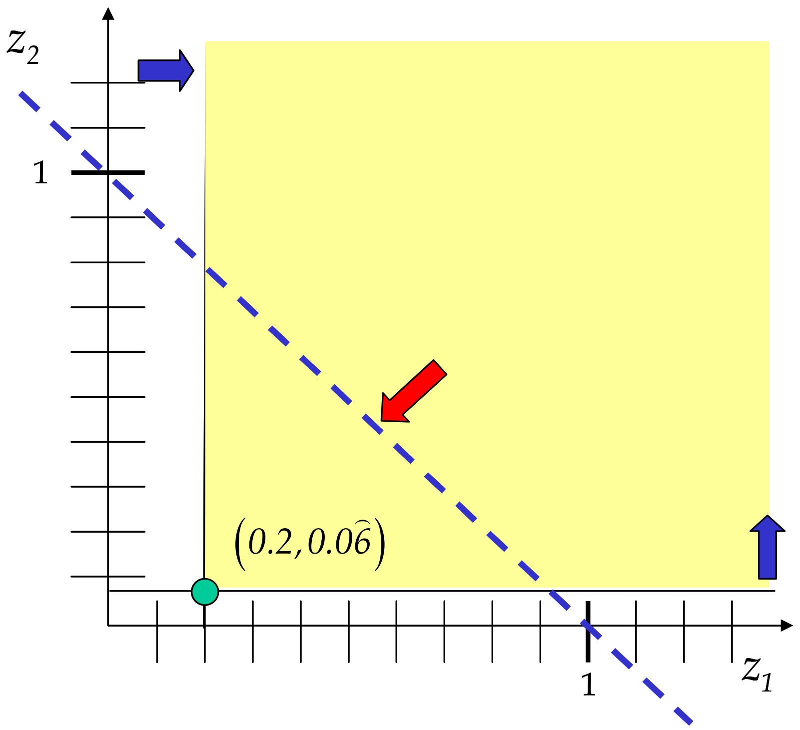
		0.25	**0.75**
		Alone on the mountain path	**With boyfriend on the easy path**
0.75	**By bike on the easy path**	*0*	*5*
0.25	**By foot on the mountain path**	*15*	*0*

Extension of solution:

When solving the linear programming problem, one can also carry out a variable transformation by dividing by the value of the game. The resulting problem from the point of view of the Wolf is:

$$\left.\begin{array}{l} max\, v \\ v \leq 15x_2 \\ v \leq 5x_1 \\ x_1 + x_2 = 1 \\ x_1, x_2 \geq 0 \end{array}\right\} \Rightarrow \begin{bmatrix} z_1 = \dfrac{x_1}{v} \\ z_2 = \dfrac{x_2}{v} \end{bmatrix} \Rightarrow \left\{\begin{array}{l} min\, z_1 + z_2 \\ 15z_2 \geq 1 \\ 5z_1 \geq 1 \\ z_1, z_2 \geq 0 \end{array}\right.$$

Solving graphically:



Once we have obtained the solution, we have to undo the variable transformation to obtain the probabilities of the optimal strategies of the Wolf.

$$z_1 + z_2 = \frac{1}{v} \Rightarrow v = \frac{1}{0.2\widehat{6}} = 3.75$$
$$x_1 = z_1 v = 0.75$$
$$x_2 = z_2 v = 0.25$$

To calculate the optimal decisions of player Little Red Riding Hood, one can consider that her linear programming model corresponds to the dual problem of Wolf's problem. Therefore, the solution to the dual problem can be obtained by calculating the relative cost coefficients of the slack variables of the primal problem, i.e., Wolf's problem.

$$\hat{C}_N = C_N - C_B B^{-1} N = (0,0) - (1,1)\begin{pmatrix} 0 & 15 \\ 5 & 0 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \left(0.0\widehat{6}\,, 0.2\right)$$

Undoing the variable transformation once we have obtained the solution of the dual, the value of the game and the optimal probabilities of player Little Red Riding Hood can be obtained as:

$$y_1 = 0.25$$
$$y_2 = 0.75$$

It can be checked that both methods yield the same result.

SOLUTION: ZERO-SUM GAME 3 X 4

a) Show that both J1 and J2 have one dominated strategy

		J2			
		I	**II**	**III**	**IV**
J1	**A**	*6*	*4*	*5*	*6*
	B	*3*	*6*	*4*	*8*
	C	*5*	*3*	*4*	*4*

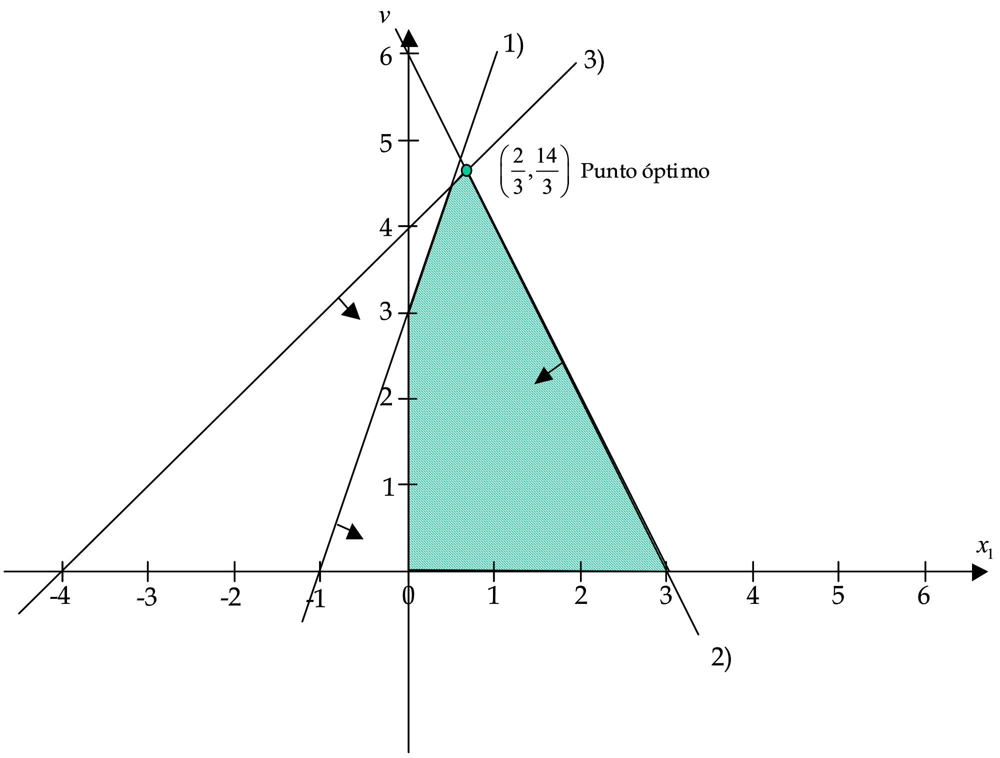
Strategy III dominates Strategy IV, and Strategy A dominates C.

b) Solve the game graphically, taking the point of view of player J1

The linear programming problem is given by:

$$\begin{array}{c} Max v \\ subject to: \\ 6x_1 + 3x_2 \geq v \\ 4x_1 + 6x_2 \geq v \\ 5x_1 + 4x_2 \geq v \\ x_1 + x_2 = 1 \end{array} \xrightarrow{x_2 = 1 - x_1} \begin{array}{c} Max v \\ s.t.: \\ 1) 3 + 3x_1 \geq v \\ 2) 6 - 2x_1 \geq v \\ 3) 4 + x_1 \geq v \end{array}$$

The graphical solution is given in the figure below:



Therefore the optimal mixed strategy $x_1 = \frac{2}{3}$ *and* $x_2 = \frac{1}{3}$ *and the value of the game is* $v = \frac{14}{3}$

c) Considering the obtained result, demonstrate that J2 never uses strategy I.

The problem of player J2 is the dual problem of the problem of J1, which we have solved graphically. The value of the variable corresponding to strategy I is zero since the relative cost coefficients of the slack variable of constraint 1) are zero because this constraint is not active in the optimal solution.

d) Show that the optimal strategy of J2 is

$$\boldsymbol{Y^* = (y_1, y_2, y_3, y_4) = \left(0, \frac{1}{3}, \frac{2}{3}, 0\right)}$$

The optimal strategy of player J2 has to yield the same objective function value as the one obtained for player J1.

$$v = \sum_i \sum_j x_i y_j \, v_{i,j} = \frac{1}{3}\frac{2}{3}4 + \frac{1}{3}\frac{1}{3}6 + \frac{2}{3}\frac{2}{3}5 + \frac{2}{3}\frac{1}{3}4 = \frac{14}{3}$$

It has to be checked that the strategy of J2 has to yield objective funciton values that are compatible with the previously obtained value, which can be seen below:

$$v \geq 6y_1 + 4y_2 + 5y_3 = 4\frac{1}{3} + 5\frac{2}{3} = \frac{13}{3}$$
$$v \geq 3y_1 + 6y_2 + 4y_3 = 6\frac{1}{3} + 4\frac{2}{3} = \frac{14}{3}$$

e) A spy offers J1 to tell him/her what J2's strategy is going to be beforehand. What is the maximum amount that J1 should be willing to pay to the spy in order for this deal to still be profitable for J1?

The maximum that player J1 can gain knowing what player J2 is going to play – taking into account that the probabilities of J2 is one-third strategy II and two-thirds strategy III:

Maximum profits with strategy II: 6

Maximum profits with strategy III: 5

Profits obtained with perfect information = $6\frac{1}{3} + 5\frac{2}{3} = \frac{16}{3}$

The value of this information is calculated by subtracting (from the profits with perfect information) the optimal value of the game, which yields:

Value of perfect information =$\frac{16}{3} - \frac{14}{3} = \frac{2}{3}$

Hence, the maximum amount that J1 would pay the spy would be 2/3.

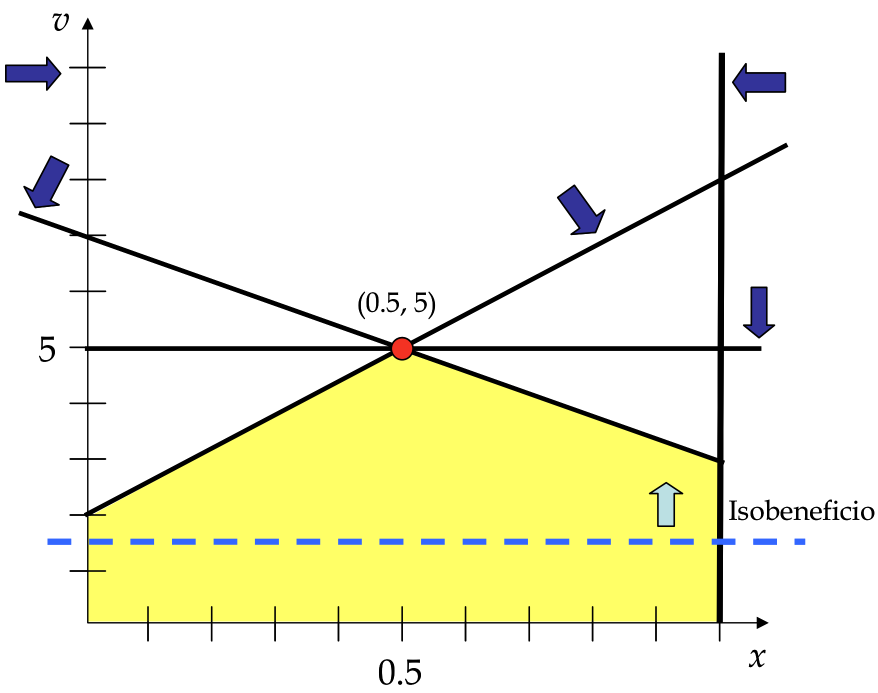
SOLUTION: ZERO-SUM GAME 2 X 4

The second strategy of J2 is dominated by the first one. There are no other dominated strategies in pure strategies for either player.

	J2		
J1	8	3	5
	2	7	5

The mixed strategy of J1 can be defined as $(x, 1-x)$ *and the formulation of the linear programming problem of J1, which will yield the optimal strategy for the game is given by:*

$$\left.\begin{array}{l} max\, v \\ 8x + 2(1-x) \geq v \\ 3x + 7(1-x) \geq v \\ 5x + 5(1-x) \geq v \\ 0 \leq x \leq 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} max\, v \\ v - 6x \leq 2 \\ v + 4x \leq 7 \\ v \leq 5 \\ 0 \leq x \leq 1 \end{array}\right.$$



The optimal solution of this problem is given by $(x, v) = (0.5,5)$. *Therefore, the optimal strategy of player J1 is to choose both strategies equally likely. The the value of the game is given by 5 units.*

From the point of view of player J2, it can be demonstrated that the third strategy takes the same value as the combination of the first strategy times 0.4 and the second times 0.6. Hence, the third strategy can be suppressed since it can be expressed as a mixed strategy of the first two pure strategies.

Therefore, the linear programming problem of player J2 can be written as:

$$\min v$$

$$8y + 3(1-y) \leq v$$

$$2y + 7(1-y) \leq v$$

$$0 \leq y \leq 1$$

This problem can be solved graphically or it can be solved substituting the value of v *in the constraints, yielding:*

$$\left.\begin{array}{l} 8y + 3(1-y) \leq 5 \\ 2y + 7(1-y) \leq 5 \end{array}\right\} \Rightarrow y = 0.4$$

Two strategies yield the same value in this game: the optimal mixed strategy of player J2 coincides with the pure strategy (that has been suppressed). Therefore, any convex linear combination of these two solutions is also optimal for this game.

- *Solution 1: (0.4, 0.6, 0)*
- *Solution 2: (0, 0, 1)*
- *Generic solution:*

$$(y_1, y_2, y_3) = \alpha(0.4,0.6,0) + (1-\alpha)(0,0,1) = (0.4\alpha, 0.6\alpha, 1-\alpha)$$
$$0 \leq \alpha \leq 1$$

Note: The solution (0,0,1) does not take advantage of the fact that player J1 might commit an error when choosing another strategy, but the optimal one. On the other hand, a mixed strategy would take advantage of this fact and result in a better solution.

<u>SOLUTION: ZERO-SUM GAME 3 X 5</u>

a) Eliminate the dominated strategies and check that the game can be reduced to a 2x3 game.

Strategy C of J2 dominates strategies B and E. After eliminating them, we obtain the matrix:

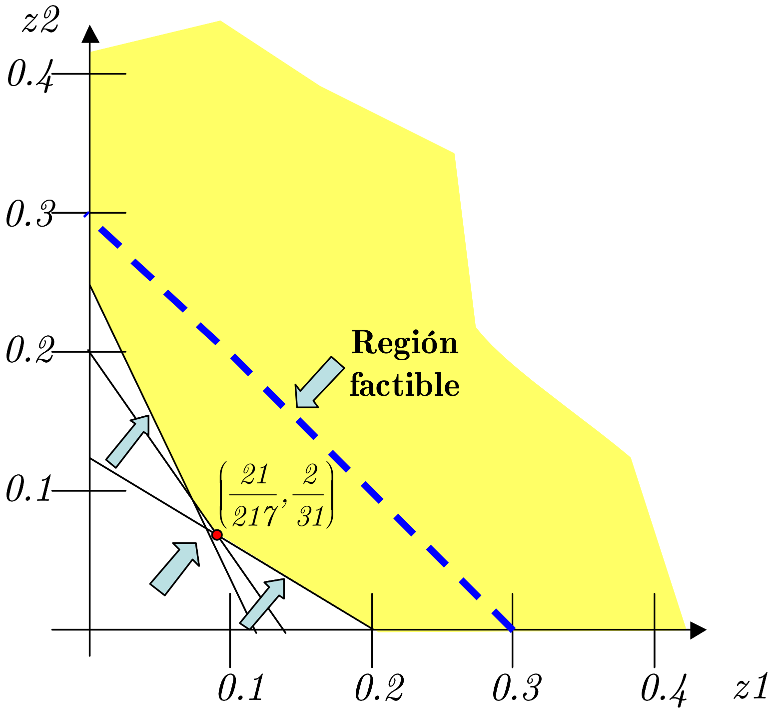
	A	*C*	*D*
1	*5*	*8*	*7*
2	*8*	*4*	*5*
3	*6*	*5*	*6*

We observe that strategy 3 of J1 is dominated by the linear convex combination (0.5, 0.5) of strategies 1 and 2, which is why strategy 3 can be eliminated. This leaves us with a 2x3 matrix.

	A	*C*	*D*
1	*5*	*8*	*7*
2	*8*	*4*	*5*

b) Solve the game graphically from the point of view of player J1

$$\left.\begin{array}{l} Max v \\ s.t.: \\ v \leq 5x_1 + 8x_2 \\ v \leq 8x_1 + 4x_2 \\ v \leq 7x_1 + 5x_2 \\ x_1 + x_2 = 1 \\ x_1, x_2 \geq 0 \end{array}\right\} \begin{array}{c} z_1 = \dfrac{x_1}{v} \\ z_2 = \dfrac{x_2}{v} \\ dividing by v \end{array} \left\{\begin{array}{l} Min z_1 + z_2 \\ s.t.: \\ 5z_1 + 8z_2 \geq 1 \\ 8z_1 + 4z_2 \geq 1 \\ 7z_1 + 5z_2 \geq 1 \\ z_1, z_2 \geq 0 \end{array}\right.$$



The optimal solution corresponds to $(z_1, z_2) = \left(\frac{21}{217}, \frac{2}{31}\right)$

The objective function value and value of the game is $v = \frac{1}{z_1+z_2} = 6.2$

The optimal probabilities of the strategies are:

$$x_1 = v \cdot z_1 = 0.6$$
$$x_2 = v \cdot z_2 = 0.4$$

c) Set up the linear programming model of J2 and starting from the solution of player J1, obtain the solution of player J2

$$\left.\begin{array}{l} Min v \\ s.t.: \\ v \geq 5y_1 + 8y_2 + 7y_3 \\ v \geq 8y_1 + 4y_2 + 5y_3 \\ y_1 + y_2 + y_3 = 1 \end{array}\right\} \begin{array}{c} w_1 = \frac{y_1}{v} \\ w_2 = \frac{y_2}{v} \\ w_3 = \frac{y_3}{v} \\ dividing by v \end{array} \left\{\begin{array}{l} Max w_1 + w_2 + w_3 \\ s.t.: \\ 5w_1 + 8w_2 + 7w_3 \leq 1 \\ 8w_1 + 4w_2 + 5w_3 \leq 1 \end{array}\right.$$

Given the graphical solution of section b), it can be observed that the second constraint is not active in the optimal solution of the problem, and therefore, the slack variable of this constraint takes a non-zero value (basic variable), and hence, its relative cost coefficient is zero, which yields (applying the duality theorem) $w_2 = 0$ *and hence* $y_2 = 0$.

The two constraints of the problem from the point of view of player J2 are fulfilled with equality which gives a value to variables w_1 *and* w_3*. Solving the system of equations yields:*

$$\left.\begin{array}{l}5w_1 + 7w_3 = 1\\8w_1 + 5w_3 = 1\end{array}\right\} w_1 = \frac{10}{155}; w_3 = \frac{3}{31}$$

Undoing the variable transformation yields:

$$y_1 = v \cdot w_1 = 0.4$$
$$y_2 = 0$$
$$y_3 = v \cdot w_3 = 0.6$$

It can be observed that the value of the game is the same from the point of view of both players.